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Wald Tests for Detecting Multiple Structural Changes in Persistence*

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Abstract

This paper considers the problem of testing for multiple structural changes in the persistence of a univariate time series. We propose sup-Wald tests of the null hypothesis that the process has an autoregressive unit root against the alternative hypothesis that the process alternates between stationary and unit root regimes. Both non-trending and trending cases are analyzed. We derive the limit distributions of the tests under the null and establish their consistency under the relevant alternatives. The computation of the test statistics as well as asymptotic critical values is facilitated by the dynamic programming algorithm proposed in Perron and Qu (2006) which allows the minimization of the sum of squared residuals under the alternative hypothesis while imposing within and cross regime restrictions on the parameters. Finally, we present Monte Carlo evidence to show that the proposed tests perform quite well in finite samples relative to those available in the literature.

Keywords: structural change, persistence, Wald tests, unit root, parameter restrictions

JEL Classification: C22

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1 Introduction

In the last twenty years or so, issues related to the detection and estimation of structural change in time series models have received a great deal of attention in both the statistics and econometrics literature (See Perron, 2006, for a survey). During this period, substantial advances have been made to cover models at a level of generality that allows a host of interesting empirical applications. These include models with general stationary regressors, models with trending variables and possible unit roots, cointegrated models and long memory processes, among others. Starting with the work of Perron (1989), a large literature has also addressed the interplay between structural changes and unit roots, in particular the fact that both classes of processes share similar qualitative features. For instance, it is now common econometric practice to test for the presence of unit roots while allowing for structural changes in the trend function of the underlying time series. The reason is that failure to account for such changes can bias unit root tests in favor of the unit root model when the true process is subject to structural changes but is otherwise (trend) stationary within regimes specified by the break dates.

The literature on testing for a change in the persistence of a time series is less extensive and, in fact, relatively recent. If such a change preserves the stationarity properties of the series in the respective regimes, methods developed in the context of stationary data can still be applied (see Andrews, 1993 and Bai and Perron, 1998). In many cases, however, a process may switch from one with an autoregressive unit root $[I(1)]$ to a stationary one $[I(0)]$ or vice-versa. This has been an issue of substantial empirical interest, especially concerning inflation rate series (e.g., Barsky, 1987, Burdekin and Siklos, 1999), short-term interest rates (e.g., Mankiw et al., 1987), government budget deficits (e.g., Hakkio and Rush, 1991) and real output (e.g., Delong and Summers, 1988). Taylor (2005) shows that standard unit root tests are not consistent against processes which display a shift in behavior from stationarity to non-stationarity and vice-versa. Hence separate methods are needed which can consistently distinguish between a process with stable persistence from processes that undergo a shift in persistence over the period under consideration.

Kim (2000), Busetti and Taylor (2004) and Harvey et al. (2006) consider testing the null hypothesis that the series is $I(0)$ throughout the sample versus the alternative that it switches from $I(0)$ to $I(1)$ and vice-versa. The tests are based on partial sums of residuals obtained by regressing the data on a constant or a constant and time trend. Leybourne et al. (2003) consider testing the null hypothesis of a stable unit root process versus the same alternatives based on the minimal value of the locally GLS detrended augmented Dickey-Fuller (ADF) unit root statistic developed in Elliott et al. (1996) over sub-samples of the data. They propose different test statistics depending on whether the initial regime is $I(1)$ or $I(0)$. When the direction of the change is unknown, they consider the minimal value of the pair of statistics for each case. Kurozumi (2005) suggests an alternative testing procedure based on the Lagrange Multiplier (LM) principle while Leybourne et al. (2006) develop tests of the unit root null based on standardized cumulative sums of squared sub-sample residuals that do not spuriously reject when the series is a constant $I(0)$ process.

The above tests are designed to detect a single change in persistence and do not allow for multiple changes. Single break tests usually have low power in detecting processes which display multiple shifts in persistence. It is thus useful to develop tests that are valid in the presence of multiple structural changes. In a recent paper, Leybourne et al. (2007) develop tests of the unit root null hypothesis based on doubly-recursive sequences of ADF-type unit root statistics and associated breakpoint estimators. Their proposed procedure can accommodate processes that exhibit multiple changes in persistence and are valid regardless of the direction of change(s). In particular, they demonstrate the consistency of their tests against such alternatives and show that their procedure can be used to consistently partition the data into its separate $I(0)$ and $I(1)$ regimes.

As is evident from this brief review, most tests for changes in persistence are based on either partial sums of the (demeaned or detrended) data or on unit root statistics applied to various data sub-samples. In contrast, this paper proposes sup-Wald tests of the null hypothesis that the process is $I(1)$ against the alternative hypothesis that the process alternates between stationary and $I(1)$ regimes. The tests are based on the difference between the sum of squared residuals from the unit root model and those from a model that allows shifts in persistence between stationary and non-stationary regimes. We consider tests for both single and multiple changes in persistence. The limit distributions of the tests are derived under the null and their consistency is established under the relevant alternatives. The computation of the test statistics as well as asymptotic critical values is facilitated by the dynamic programming algorithm proposed in Perron and Qu (2006) which allows the minimization of the sum of squared residuals under the alternative hypothesis while imposing within and cross regime restrictions on the parameters. Finally, we present Monte Carlo evidence to show that the proposed tests perform quite well in finite samples relative to those proposed in Leybourne et al. (2007).

The paper is organized as follows. Section 2 presents the models and the test statistics.

In Section 3, we discuss issues related to the computation of the statistics with reference to the dynamic programming algorithm proposed in Perron and Qu (2006). Section 4 details the asymptotic properties of the tests under the null and alternative hypotheses. Monte Carlo simulations are presented in Section 5 to assess the adequacy of the asymptotic approximations in finite samples. Some recommendations for applied work are also included. Section 6 concludes. All technical derivations are included in a mathematical appendix.

2 The Models and Test Statistics

Consider a scalar random variable y_t generated by

$$y_t = c_i + \alpha_i y_{t-1} + u_t \tag{1}$$

for $t \in [T_{i-1}+1, T_i],\ i = 1, ..., m+1$, where we use the convention $T_0 = 0$ and $T_{m+1} = T$, with T denoting the sample size. The vector of break fractions is denoted $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_i = T_i/T$ for $i = 1, ...m$. The errors $\{u_t\}$ are generated by the stationary linear process

$$u_t = d(L)v_t,\ d(L) = \sum_{s=0}^{\infty} d_s L^s \tag{2}$$

where $\sum_{s=1}^{\infty} s\,|d_s| < \infty$. Also, α_i should be understood as standing for the sum of the coefficients in the autoregressive representation for y_t in regime i. We make the following assumption regarding the innovation process $\{v_t\}$:

Assumption A1: The process $\{v_t\}$ is a martingale difference sequence with $E(v_t^2|v_{t-1}, ...) = \sigma^2,\ E(|v_t|^r\,|v_{t-1}, ...) = \kappa_r\ (r = 3, 4)$ and $\sup_t\ E(|v_t|^{4+\beta}\,|v_{t-1}, ...) = \kappa < \infty$ for some $\beta > 0$.

Without loss of generality, we assume that the initial values $y_0 = u_0 = 0$. Next, we make the following assumption regarding the polynomial $d(L)$:

Assumption A2: All roots of $d(L)$ are outside the unit circle.

We consider the following two models depending on whether the initial regime contains a unit root or not:

- Model 1a: $c_i = 0,\ \alpha_i = 1$ in odd regimes and $|\alpha_i| < 1$ in even regimes.
- Model 1b: $c_i = 0,\ \alpha_i = 1$ in even regimes and $|\alpha_i| < 1$ in odd regimes.

In model 1a, the process alternates between a unit root and a stationary process with a unit root in the first regime. Model 1b is similar except that the first regime is stationary. To allow for the possibility of trending data, we also consider the process

$$y_t = c_i + b_i t + \alpha_i y_{t-1} + u_t$$

The corresponding models are

- Model 2a: $\alpha_i = 1,\ b_i = 0$ in odd regimes and and $|\alpha_i| < 1$ in even regimes.
- Model 2b: $\alpha_i = 1,\ b_i = 0$ in even regimes and $|\alpha_i| < 1$ in odd regimes.

We are interested in testing the null hypothesis that y_t is $I(1)$ throughout the sample. In the context of models 1a and 1b, this implies H_0: $c_i = 0,\ \alpha_i = 1$ for all i. For models 2a and 2b, the null hypothesis is H_0: $c_i = c,\ b_i = 0,\ \alpha_i = 1$ for all i. We first consider the test statistics for non-trending data, i.e., those based on models 1a and 1b. Under (1) and Assumption A2, y_t evolves according to

$$\Delta y_t = c_i + (\alpha_i - 1)y_{t-1} + \sum_{j=1}^{\infty} \pi_j u_{t-j} + v_t$$

where the coefficients π_j $(j = 1, ..., \infty)$ are functions of the parameters $d_s,\ s \geq 0$. Since $\Delta y_t = u_t$ under the null hypothesis, we have the representation

$$\Delta y_t = c_i + (\alpha_i - 1)y_{t-1} + \sum_{j=1}^{\infty} \pi_j \Delta y_{t-j} + v_t$$

We can approximate this infinite autoregression by a truncated version whose order is a function of the sample size T:

$$\Delta y_t = c_i + (\alpha_i - 1)y_{t-1} + \sum_{j=1}^{l_T} \pi_j \Delta y_{t-j} + v_t^* \tag{3}$$

where $v_t^* = \sum_{j=l_T+1}^{\infty} \pi_j \Delta y_{t-j} + v_t$.

Note that the coefficients π_j on the lagged first-differences are not allowed to change under both the null and alternative hypotheses. Allowing them to change across regimes would open up the possibility that the tests reject because of changes in the short-run dynamics instead of the $I(0)/I(1)$ nature of the process. Hence, the need to constrain them to be fixed. We shall, however, show via simulations that the exact sizes of our tests are quite robust when the process is $I(1)$ throughout with changes in the short-run dynamics. Though this restriction is not imposed by Leybourne et al. (2007), as we shall see the size of their test is quite sensitive to variations in the short-run dynamics. It is important, however, to allow the constant to change across regimes when the process is $I(0)$. This is because a change from an $I(1)$ to an $I(0)$ process is often accompanied with a change in the long-run mean of the process due to the fact that the level of the series in an $I(1)$ regime tends to wander arbitrarily as opposed to what occurs in an $I(0)$ regime for which the series tends to a stable trend path.

We study three types of tests. First, we consider the Wald test that applies when the alternative involves a fixed value $m = k$ of changes. For models 1a and 1b, the test is defined as

$$\begin{aligned} F_{1a}(\lambda, k) &= \frac{(T-k-l_T)(SSR_0 - SSR_{1a,k})}{kSSR_{1a,k}} \quad \text{if } k \text{ is even} \\ F_{1a}(\lambda, k) &= \frac{(T-k-1-l_T)(SSR_0 - SSR_{1a,k})}{(k+1)SSR_{1a,k}} \quad \text{if } k \text{ is odd} \end{aligned} \tag{4}$$

$$\begin{aligned} F_{1b}(\lambda, k) &= \frac{(T-k-2-l_T)(SSR_0 - SSR_{1b,k})}{(k+2)SSR_{1b,k}} \quad \text{if } k \text{ is even} \\ F_{1b}(\lambda, k) &= \frac{(T-k-1-l_T)(SSR_0 - SSR_{1b,k})}{(k+1)SSR_{1b,k}} \quad \text{if } k \text{ is odd} \end{aligned} \tag{5}$$

In (4) and (5), SSR_0 denotes the sum of squared residuals under the null hypothesis, i.e. the sum of squared residuals obtained estimating (3) by OLS subject to the restrictions $c_i = 0,\ \alpha_i = 1$ for all i. The quantity $SSR_{k,1a}$ denotes the sum of squared residuals obtained from estimating (3) under the restrictions imposed by Model 1a. Similarly, $SSR_{k,1b}$ denotes the sum of squared residuals obtained from estimating (3) under the restrictions imposed by Model 1b. Next, we define the following set for some arbitrary small positive number ϵ: $\Lambda_\epsilon^k = \{\lambda : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1-\epsilon\}$. The sup-Wald tests are then defined as $\sup F_{1a}(k) = \sup_{\lambda \in \Lambda_\epsilon^k} F_{1a}(\lambda, k)$ and $\sup F_{1b}(k) = \sup_{\lambda \in \Lambda_\epsilon^k} F_{1b}(\lambda, k)$. Since the estimates $\widetilde{\lambda} = \{\widetilde{\lambda}_1, ..., \widetilde{\lambda}_k\}$ with $\widetilde{\lambda}_i = \widetilde{T}_i/T$ (for $i = 1, ..., k$) obtained by minimizing the global sum of squared residuals correspond to those that maximize the Wald test, we have $\sup F_{1a}(k) = F_{1a}(\widetilde{\lambda}, k)$ and $\sup F_{1b}(k) = F_{1b}(\widetilde{\lambda}, k)$. Note that to ensure that the Wald tests are non-negative in finite samples, the same number of lags of first differences of the dependent variable must be used when estimating the models under the null and alternative hypotheses.

The second procedure applies when the alternative hypothesis involves an unknown number of changes between 1 and some upper bound, say A. As in Bai and Perron (1998), we consider a double maximum test based on the maximum of the individual tests for the null of no break versus m breaks ($m = 1, ..., A$), defined by

$$\begin{aligned} UDmax_{1a}(A) &= \max_{1 \le m \le A} \sup_{\lambda \in \Lambda_\epsilon^m} F_{1a}(\lambda, m), \\ UDmax_{1b}(A) &= \max_{1 \le m \le A} \sup_{\lambda \in \Lambda_\epsilon^m} F_{1b}(\lambda, m). \end{aligned}$$

This test is useful when the number of breaks is unknown. The third type of tests is based on the presumption that the nature of persistence in the first regime is unknown, i.e., we do not have any a priori knowledge regarding whether the first regime contains a unit root or not. The tests are given by

$$\begin{aligned} W_1(k) &= \max[\sup F_{1a}(\lambda, k), \sup F_{1b}(\lambda, k)] \\ Wmax_1 &= \max_{1 \le m \le A} W_1(m) \end{aligned}$$

For models 2a and 2b, regression (3) is replaced by

$$\Delta y_t = c_i + b_i t + (\alpha_i - 1) y_{t-1} + \sum_{j=1}^{l_T} \pi_j \Delta y_{t-j} + v_t^* \tag{6}$$

The Wald tests are defined as

$$\begin{aligned} F_{2a}(\lambda, k) &= \frac{(T - 2k - 1 - l_T)(SSR_0^* - SSR_{2a,k})}{(2k) SSR_{2a,k}} \quad \text{if } k \text{ is even} \\ F_{2a}(\lambda, k) &= \frac{(T - 2k - 2 - l_T)(SSR_0^* - SSR_{2a,k})}{(2k+1) SSR_{2a,k}} \quad \text{if } k \text{ is odd} \end{aligned} \tag{7}$$

$$\begin{aligned} F_{2b}(\lambda, k) &= \frac{(T - 2k - 3 - l_T)(SSR_0^* - SSR_{2b,k})}{(2k+2) SSR_{2b,k}} \quad \text{if } k \text{ is even} \\ F_{2b}(\lambda, k) &= \frac{(T - 2k - 2 - l_T)(SSR_0^* - SSR_{2b,k})}{(2k+1) SSR_{2b,k}} \quad \text{if } k \text{ is odd} \end{aligned} \tag{8}$$

In (7) and (8), SSR_0^* denotes the sum of squared residuals under the null hypothesis, i.e. the sum of squared residuals obtained estimating (6) subject to the restrictions $c_i = c$, $b_i = 0$, $\alpha_i = 1$ for all i. Given these tests, the remaining statistics are defined in the same way as for models 1a and 1b. These are denoted sup $F_{2a}(k)$, sup $F_{2b}(k)$, $UDmax_{2a}(A)$, $UDmax_{2b}(A)$, $W_2(k)$ and $Wmax_2$.

3 Computing the Test Statistics

In order to compute the sup-Wald test for any particular model, we need to minimize the global sum of squared residuals over the set of permissible break fractions Λ_ϵ^k subject to the restrictions implied by the model. Note that there are two types of restrictions: one is model-specific which involves imposing unit roots within the relevant regimes while the other ensures that the coefficients of the lagged first differences do not change across regimes. Note that our procedure does not impose the stationarity restrictions ($|\alpha_i| < 1$). While it may be desirable to impose these restrictions, it will make little difference in practice given that explosive alternatives with $|\alpha_i| > 1$ are unlikely to arise in practice.

Bai and Perron (2003) describe an efficient estimation procedure based on a dynamic programming algorithm which involves at most least-squares operations of order $O(T^2)$ for any number of breaks. However, their procedure is not directly applicable in our context since it is does not account for parametric restrictions within and across regimes. Building on the work of Bai and Perron (2003), Perron and Qu (2006) develop a recursive procedure that allows the minimization of sum of squared residuals in general multiple structural change models subject to restrictions. We first describe their framework and subsequently discuss how the models considered in this paper can be expressed as special cases.

Perron and Qu (2006) consider a multiple linear regression model with k breaks or $k+1$ regimes. Let $y = (y_1, ..., y_T)'$ be the dependent variable and $Z = (z_1, ..., z_T)'$ be a T by q matrix of regressors. Define $\bar{Z} = diag(Z_1, ..., Z_{k+1})$, a T by $(k+1)q$ matrix with $Z_i = (z_{T_{i-1}+1}, ..., z_{T_i})'$ for $i = 1, ..., k+1$. The matrix $\bar{Z}$ is the diagonal partition of Z at the set of break points $(T_1, ..., T_k)$. The $(k+1)q$ vector of coefficients is $\delta = (\delta_1', ..., \delta_{k+1}')'$. The general pure structural change model with restrictions on the coefficients can be expressed as

$$y = \bar{Z}\delta + u \tag{9}$$

where

$$R\delta = r \tag{10}$$

with R a s by $(k+1)q$ matrix with rank s and r a s dimensional vector of constants. The estimated break dates are obtained as $(\widetilde{T}_1, ..., \widetilde{T}_k) = \arg\min_{T_1,...,T_k} SSR^R(T_1, ..., T_k)$ where $SSR^R(T_1, ..., T_k)$ is the sum of squared residuals from the restricted OLS regression evaluated at the partition $\{T_1, ..., T_k\}$. Details on the recursive procedure can be found in Section 5.2 of Perron and Qu (2006).

The models described in Section 2 can be obtained as special cases of the framework represented by (9) and (10). For all models, r is a zero vector of dimension given by the number of restrictions. We illustrate the form of the R matrix for models 1a and 2a. First, consider model 1a. We have $z_t = (1, y_{t-1}, \Delta y_{t-1}, ..., \Delta y_{t-l_T})'$, $\delta_i = (c_i, \alpha_i - 1, \pi_{1i}, ..., \pi_{ji})'$. With k even, R is a $[k+2+kl_T]$ by $[(l_T+2)(k+1)]$ matrix where the first $k+2$ restrictions are implied by the unit roots imposed in the $(k/2+1)$ odd regimes and the last kl_T restrictions are implied by the fact that the coefficients $\pi_1, ..., \pi_{l_T}$ are fixed across regimes. For instance, with $k = l_T = 2$, R is the 8 by 12 matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Similarly, when k is odd, R is a $[k+1+kl_T]$ by $[(l_T+2)(k+1)]$ matrix where the first $k+1$ restrictions are implied by the unit roots imposed in the $(k+1)/2$ even regimes and the last kl_T restrictions again follow from the constancy of the coefficients of the lagged first differences of the dependent variable.

For model 2a, we have $z_t = (1, y_{t-1}, t, \Delta y_{t-1}, ..., \Delta y_{t-l_T})'$, $\delta_i = (c_i, b_i, \alpha_i - 1, \pi_{1i}, ..., \pi_{ji})'$. Here the zero restrictions on the intercept are replaced by zero restrictions on the trend coefficients. With k even, R is a $[k+2+kl_T]$ by $[(l_T+3)(k+1)]$ matrix and with k odd, R is a $[k+1+kl_T]$ by $[(l_T+3)(k+1)]$ matrix. For instance, with $k = l_T = 2$, R is the 8 by 15 matrix

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We can similarly express the restrictions implied by the other models in terms of the general model considered in Perron and Qu (2006). We can thus directly apply their algorithm to minimize the sum of squared residuals subject to the relevant restrictions.

4 Asymptotic Results

This section details the limiting properties of the proposed statistics under the null and alternative hypotheses. Specifically, in subsection 4.1, we present the asymptotic distributions of the tests under the null hypothesis that the process is $I(1)$ throughout the sample. The computation of asymptotic critical values is discussed in subsection 4.2. Finally, in subsection 4.3, we demonstrate the consistency of the tests under the relevant alternative hypotheses.

4.1 The Null Limiting Distributions

Let $W(.)$ denote a standard Brownian motion on $[0,1]$. Also, let $W^{(j)}(r)$ and $\widetilde{W}^{(j)}(r)$ represent demeaned and detrended Brownian motions respectively, over $r \in (\lambda_{j-1}, \lambda_j)$ (the appendix contains expressions for these in terms of the standard Wiener process $W(.)$). The following theorem states the limit distributions of the tests under the null hypothesis of a unit root. We start with the case where there is no serial correlation, i.e., $u_t = v_t$ and subsequently show that all limit results are valid for the general case.

Theorem 1 *Assume that $u_t = v_t$ where v_t satisfies Assumption A2. Suppose also that the test statistics are constructed based on autoregressions that do not include the lags of first differences of y_t. Then under the null hypothesis H_0: $c_i = 0,\ \alpha_i = 1$ for all i, if k is even, we have*

$$F_{1a}(\lambda, k) \ \Rightarrow\ \frac{1}{k}\sum_{i=1}^{k/2}\left[\frac{\left\{\int_{\lambda_{2i-1}}^{\lambda_{2i}} W^{(2i)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}[W^{(2i)}(r)]^2 dr} + \frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i}) - W(\lambda_{2i-1})\right\}^2\right]$$

$$F_{1b}(\lambda, k) \ \Rightarrow\ \frac{1}{k+2}\sum_{i=0}^{k/2}\left[\frac{\left\{\int_{\lambda_{2i}}^{\lambda_{2i+1}} W^{(2i+1)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i}}^{\lambda_{2i+1}}[W^{(2i+1)}(r)]^2 dr} + \frac{1}{\lambda_{2i+1}-\lambda_{2i}}\left\{W(\lambda_{2i+1}) - W(\lambda_{2i})\right\}^2\right]$$

If k is odd,

$$F_{1a}(\lambda, k) \Rightarrow \frac{1}{k+1}\sum_{i=1}^{(k+1)/2}\left[\frac{\left\{\int_{\lambda_{2i-1}}^{\lambda_{2i}} W^{(2i)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}[W^{(2i)}(r)]^2 dr} + \frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i}) - W(\lambda_{2i-1})\right\}^2\right]$$

$$F_{1b}(\lambda,k) \Rightarrow \frac{1}{k+1}\sum_{i=0}^{(k-1)/2}\left[\frac{\left\{\int_{\lambda_{2i}}^{\lambda_{2i+1}} W^{(2i+1)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i}}^{\lambda_{2i+1}}[W^{(2i+1)}(r)]^2dr}+\frac{1}{\lambda_{2i+1}-\lambda_{2i}}\left\{W(\lambda_{2i+1})-W(\lambda_{2i})\right\}^2\right]$$

Under the null hypothesis H_0: $c_i = c$, $b_i = 0$, $\alpha_i = 1$ for all i, if k is even, we have

$$F_{2a}(\lambda,k) \;\Rightarrow\; \frac{1}{2k}\left[\begin{array}{c} -\{W(1)\}^2+\sum_{i=0}^{k/2}\left[\frac{1}{\lambda_{2i+1}-\lambda_{2i}}\left\{W(\lambda_{2i+1})-W(\lambda_{2i})\right\}^2\right] \\ +\sum_{i=1}^{k/2}\left[\begin{array}{c}\frac{\left\{\int_{\lambda_{2i-1}}^{\lambda_{2i}}\widetilde{W}^{(2i)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}[\widetilde{W}^{(2i)}(r)]^2dr}+\frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i})-W(\lambda_{2i-1})\right\}^2 \\ +\frac{\left[\int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}}rdr\}dW(r)\right]^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}}rdr\}^2dr}\end{array}\right]\end{array}\right]$$

$$F_{2b}(\lambda,k) \;\Rightarrow\; (2k+2)^{-1}\left[\begin{array}{c} -W(1)^2+\sum_{i=1}^{k/2}\left[\frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i})-W(\lambda_{2i-1})\right\}^2\right] \\ +\sum_{i=0}^{k/2}\left[\begin{array}{c}\frac{\left\{\int_{\lambda_{2i}}^{\lambda_{2i+1}}\widetilde{W}^{(2i+1)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i}}^{\lambda_{2i+1}}[\widetilde{W}^{(2i+1)}(r)]^2dr}+\frac{1}{\lambda_{2i+1}-\lambda_{2i}}\left\{W(\lambda_{2i+1})-W(\lambda_{2i})\right\}^2 \\ +\frac{\left[\int_{\lambda_{2i}}^{\lambda_{2i+1}}\{r-(\lambda_{2i+1}-\lambda_{2i})^{-1}\int_{\lambda_{2i}}^{\lambda_{2i+1}}rdr\}dW(r)\right]^2}{\int_{\lambda_{2i}}^{\lambda_{2i+1}}\{r-(\lambda_{2i+1}-\lambda_{2i})^{-1}\int_{\lambda_{2i}}^{\lambda_{2i+1}}rdr\}^2dr}\end{array}\right]\end{array}\right]$$

If k is odd,

$$F_{2a}(\lambda,k) \;\Rightarrow\; \frac{1}{2k+1}\left[\begin{array}{c} -\{W(1)\}^2+\sum_{i=0}^{(k-1)/2}\left[\frac{1}{\lambda_{2i+1}-\lambda_{2i}}\left\{W(\lambda_{2i+1})-W(\lambda_{2i})\right\}^2\right] \\ +\sum_{i=1}^{(k+1)/2}\left[\begin{array}{c}\frac{\left\{\int_{\lambda_{2i-1}}^{\lambda_{2i}}\widetilde{W}^{(2i)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}[\widetilde{W}^{(2i)}(r)]^2dr}+\frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i})-W(\lambda_{2i-1})\right\}^2 \\ +\frac{\left[\int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}}rdr\}dW(r)\right]^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}}rdr\}^2dr}\end{array}\right]\end{array}\right]$$

$$F_{2b}(\lambda,k) \;\Rightarrow\; \frac{1}{2k+1}\left[\begin{array}{c} -\{W(1)\}^2+\sum_{i=1}^{(k+1)/2}\left[\frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i})-W(\lambda_{2i-1})\right\}^2\right] \\ +\sum_{i=1}^{(k-1)/2}\left[\begin{array}{c}\frac{\left\{\int_{\lambda_{2i}}^{\lambda_{2i+1}}\widetilde{W}^{(2i+1)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i}}^{\lambda_{2i+1}}[\widetilde{W}^{(2i+1)}(r)]^2dr}+\frac{1}{\lambda_{2i+1}-\lambda_{2i}}\left\{W(\lambda_{2i+1})-W(\lambda_{2i})\right\}^2 \\ +\frac{\left[\int_{\lambda_{2i}}^{\lambda_{2i+1}}\{r-(\lambda_{2i+1}-\lambda_{2i})^{-1}\int_{\lambda_{2i}}^{\lambda_{2i+1}}rdr\}dW(r)\right]^2}{\int_{\lambda_{2i}}^{\lambda_{2i+1}}\{r-(\lambda_{2i+1}-\lambda_{2i})^{-1}\int_{\lambda_{2i}}^{\lambda_{2i+1}}rdr\}^2dr}\end{array}\right]\end{array}\right]$$

Theorem 1 shows that for all models, the limit distributions of the Wald tests based on a given vector of break fractions $(\lambda_1, ..., \lambda_k)$ are pivotal and depend only on functionals of a Wiener processes. The limit distributions are different depending on whether the alternative hypothesis specifies that the initial regime has a unit root or is stationary. As is the case with standard unit root tests, the limits are also different for the trending and non-trending cases. The form of the distributions vary according to whether the number of breaks under the alternative hypothesis is even or odd. With these theoretical results, we can obtain the limit distributions of the proposed tests as a direct consequence of the continuous mapping theorem. These are stated in the following Corollary.

Corollary 1 *Denote the limit distribution of the test* $F_j(\lambda, k)$ *by* $F_j^*(\lambda, k)$, $j = 1a, 1b, 2a, 2b$. *Then, under the same null hypothesis as in Theorem 1, we have: a)* $\sup_{\lambda \in \Lambda_\epsilon^k} F_j(\lambda, k) \Rightarrow \sup_{\lambda \in \Lambda_\epsilon^k} F_j^*(\lambda, k)$, *b)* $UDmax_j(A) \Rightarrow \max_{1 \leq m \leq A} \sup_{\lambda \in \Lambda_\epsilon^m} F_j^*(\lambda, m)$, *c)* $W_1(k) \Rightarrow \max[\sup_{\lambda \in \Lambda_\epsilon^k} F_{1a}^*(\lambda, k), \sup_{\lambda \in \Lambda_\epsilon^k} F_{1b}^*(\lambda, k)]$, $W_2(k) \Rightarrow \max[\sup_{\lambda \in \Lambda_\epsilon^k} F_{2a}^*(\lambda, k), \sup_{\lambda \in \Lambda_\epsilon^k} F_{2b}^*(\lambda, k)]$, *d)* $Wmax_1 \Rightarrow \max_{1 \leq m \leq A}[\max[\sup_{\lambda \in \Lambda_\epsilon^m} F_{1a}^*(\lambda, m), \sup_{\lambda \in \Lambda_\epsilon^m} F_{1b}^*(\lambda, m)]]$, $Wmax_2 \Rightarrow \max_{1 \leq m \leq A}[\max[\sup_{\lambda \in \Lambda_\epsilon^m} F_{2a}^*(\lambda, m), \sup_{\lambda \in \Lambda_\epsilon^m} F_{2b}^*(\lambda, m)]]$.

Next, we show that the results of Theorem 1 and Corollary 1 remain valid when u_t follows the general linear process specified by (2). We make the following assumption regarding the lag length l_T.

Assumption A3: As $T \to \infty$, the lag length l_T is assumed to satisfy (a) (upper bound condition) $l_T^2/T \to 0$ and (b) (lower bound condition) $l_T \sum_{j > l_T} \pi_j \to 0$.

The implication of the lower bound condition in practice is that it allows for a logarithmic rate of increase for l_T thereby allowing for the use of data dependent rules such as information criteria to select the lag length (see Ng and Perron, 1995). We now state the result for the general case.

Theorem 2 *Under A1-A3 hold and the null hypotheses considered in Theorem 1, the corresponding test statistics have the same limit distributions as those stated in Theorem 1 and Corollary 1.*

4.2 Asymptotic Critical Values

Given the non-standard nature of the limit distributions, the critical values are obtained by Monte-Carlo simulations. Here again we use Perron and Qu's (2006) dynamic programming algorithm. First, we generate a sample of $T = 500$ observations from a random walk with i.i.d. $N(0, 1)$ errors. We then apply the algorithm to obtain the minimized sum of squared residuals and the corresponding vector of break fractions subject to the relevant restrictions. Next, we simulate a Wiener process using the partial sums of 500 i.i.d. $N(0, 1)$ random variables. Finally, we evaluate the expressions appearing in the limit distributions (see Appendix) at the vector of break fractions obtained earlier. This procedure is repeated 5000 times to obtain the required quantiles of the limit distributions.

Asymptotic critical values are provided in Table 1 with the level of trimming set at $\epsilon = 0.15$. The maximum number of breaks considered is 5. Panel A provides critical values for the non-trending case while those for the trending case are presented in Panel B. The critical values for models 1a and 2a are larger than those for models 1b and 2b respectively. Note also that the critical values are not monotonically decreasing as k increases. This is due to the fact that the limit distributions are different for the cases with k even or odd. For even or odd values they are monotonically decreasing as expected.

4.3 Consistency

In this subsection, we study the properties of the tests under the relevant alternative hypotheses. In particular, we demonstrate that in the presence of regime shifts in persistence of the form considered in this paper, the relevant tests are consistent, i.e., they reject the null hypothesis with probability one in large samples. We make the following assumption regarding the location of the true break fractions.

Assumption A4: The true vector of break fractions, denoted $\lambda^0 = (\lambda_1^0, ..., \lambda_m^0)$, is assumed to belong to the set of permissible break fractions, i.e., $\lambda^0 \in \Lambda_\epsilon^m$.

This assumption is not very restrictive given that in practice, ϵ can be chosen to be small. We can then state the following theorem regarding the consistency of the tests under the relevant alternative hypotheses.

Theorem 3 *Suppose that the data are generated under the alternative hypothesis represented by model j ($j = 1a, 1b, 2a$ or $2b$) with m breaks in persistence. Then under A1-A4, the tests $\sup_{\lambda \in \Lambda_\epsilon^m} F_j(\lambda, m)$ and $UDmax_j(A)$ are consistent. Moreover, if the data are generated by models 1a or 1b, the tests $W_1(m)$ and $Wmax_1$ are consistent while if the data are generated by models 2a or 2b, the tests $W_2(m)$ and $Wmax_2$ are consistent.*

5 Simulation Experiments

In this section, we conduct simulation experiments to assess the finite sample performance of the proposed tests as well as to provide a comparison with the tests proposed in Leybourne et al. (2007). The latter class of tests is based on a doubly-recursive application of the unit root statistic using the local GLS detrending methodology developed in Elliott et al (1996).

More specifically, Leybourne et al. (2007) propose the test statistic

$$M = \inf_{\lambda \in (0,1)} \inf_{\tau \in (\lambda,1]} DF_G(\lambda, \tau)$$

where $DF_G(\lambda, \tau)$ is the local GLS detrended ADF unit root t-statistic that uses the observations between λT and τT. They derive the limit distribution of M for both trending and non-trending cases and demonstrate that the test is consistent against multiple changes in persistence, irrespective of whether the initial regime has a unit root or not.

For our Monte-Carlo exercise, we consider cases where the data generating processes (DGPs) involve no break (size) as well as those that involve one and two breaks (power). Results are presented for models 1a and 1b. Those for models 2a and 2b are qualitatively similar and hence not reported. The sample sizes used are $T = 150, 240$. The maximum number of allowable breaks is set at five. The lag length in the autoregression is selected using the Bayesian Information Criterion with the maximum number of lags allowed set at ten. In our simulation experiments, we first obtain the number of lags based on the estimation of the alternative model and then use this number in the estimation of the null model. In all experiments, $\{e_t\}$ denotes a sequence of $i.i.d.$ $N(0,1)$ variables. The errors $\{u_t\}$ are generated by the ARMA process

$$u_t = \rho u_{t-1} + e_t + \theta e_{t-1}, \; u_0 = 0 \tag{11}$$

We present results for the following combinations of values of the autoregressive parameter (ρ) and the moving average parameter (θ): (a) $\rho = \theta = 0$, (b) $\rho = 0.3,\ \theta = 0$, (c) $\rho = 0.5,\ \theta = 0$, (d) $\rho = 0,\ \theta = 0.5$, (e) $\rho = 0,\ \theta = -0.5$, (f) $\rho = 0.3,\ \theta = 0.5$, (h) $\rho = 0.3,\ \theta = -0.5$. The nominal size for all tests is set at 5%. All experiments are based on 1000 replications.

5.1 The Empirical Size of the Tests

In order to assess the empirical size of the tests, the DGP considered is

- DGP-0:

$$\Delta y_t = \pi_1 \Delta y_{t-1} + u_t \quad \text{if} \quad t \leq [0.5T]$$
$$\Delta y_t = \pi_2 \Delta y_{t-1} + u_t \quad \text{if} \quad t \geq [0.5T] + 1$$

  with $y_0 = 0$.

The base case to be analyzed is $\pi_1 = \pi_2 = 0$. However, given that our regression model constrains the parameters governing the short-run dynamics to remain the same across regimes, we also present some results for cases where $\pi_1 \neq \pi_2$ to investigate the effect of unstable short-run parameters on the empirical size of the tests. Table 2.1 presents results for the case $\pi_1 = \pi_2 = 0$. First, when the errors do not contain a MA component, all the proposed statistics are adequately sized with the null rejection probabilities never exceeding 10% for either sample size. When a positive MA component is present, the $UDmax_{1b}$ test is somewhat over-sized with $T = 150$ but the distortions diminish when the sample size is increased. With a negative MA component, however, the over-sizing problem is more severe and, in some cases (especially for test statistics based on Model 1b), the distortions remain prominent even for $T = 240$. As with standard unit root tests, these size problems arise from the downward bias in the persistence parameter estimates under the null hypothesis of a unit root.

The M test, on the other hand, is seriously over-sized irrespective of the nature and extent of serial correlation in the errors. The rejection probability is at least 15% for $T = 150$ and never falls below 10% even for $T = 240$. These distortions are especially severe (much more so compared to the proposed tests) with negative MA errors. For instance, with $\rho = 0$, $\theta = -0.5$ and $T = 240$, the M test rejects the null hypothesis in 83% of the samples. Since the M test is based on the application of unit root tests to data sub-samples, the bias in the autoregressive parameter estimates is exacerbated which in turn contributes to the poor finite sample performance of the test under the null hypothesis.

Table 2.2 reports the rejection frequencies when $\pi_1 \neq \pi_2$ and $\rho = \theta = 0$. The proposed tests have empirical sizes that are generally greater than the ones with $\pi_1 = \pi_2 = 0$ although the magnitude of the distortions is not substantial. In contrast, the null rejection probabilities of the M test increase quite sharply relative to the case where the DGP does not involve a shift in short-run dynamics across regimes.

5.2 The Case with One Break

Here we provide a power comparison of the various tests when the DGPs involve a single break in persistence. We consider the following DGPs:

- DGP-1:

$$\begin{aligned} y_t &= y_{t-1} + u_t \quad \text{if} \quad t \leq [T\lambda_1^0] \\ y_t &= \alpha y_{t-1} + u_t \quad \text{if} \quad t \geq [T\lambda_1^0] + 1 \end{aligned}$$

- DGP-2:

$$y_t = \alpha y_{t-1} + u_t \quad \text{if} \quad t \leq [T\lambda_1^0]$$
$$y_t = y_{t-1} + u_t \quad \text{if} \quad t \geq [T\lambda_1^0] + 1$$

- DGP-3:

$$y_t = y_{t-1} + \pi_1 \Delta y_{t-1} + e_t \quad \text{if} \quad t \leq [T\lambda_1^0]$$
$$y_t = \alpha y_{t-1} + \pi_2 \Delta y_{t-1} + e_t \quad \text{if} \quad t \geq [T\lambda_1^0] + 1$$

- DGP-4:

$$y_t = \alpha y_{t-1} + \pi_1 \Delta y_{t-1} + e_t \quad \text{if} \quad t \leq [T\lambda_1^0]$$
$$y_t = y_{t-1} + \pi_2 \Delta y_{t-1} + e_t \quad \text{if} \quad t \geq [T\lambda_1^0] + 1$$

- DGP-5:

$$y_t = y_{t-1} + u_t \quad \text{if} \quad t \leq [T\lambda_1^0]$$
$$y_t - y_{[T\lambda_1^0]} = \alpha(y_{t-1} - y_{[T\lambda_1^0]}) + u_t \quad \text{if} \quad t \geq [T\lambda_1^0] + 1$$

DGP-1 and DGP-2 are processes which involve a shift in the persistence parameter but no change in the short-run dynamics across regimes. DGP-3 and DGP-4 allow for the short-run dynamics to simultaneously change as well. DGP-5 is a variant of DGP-1 that is considered in Leybourne et al. (2007). Such a process is designed to avoid sharp jumps to zero at the break point between the $I(1)$ and $I(0)$ regimes and ensures a joining up of these regimes. We consider three values for the location of the break: $\lambda_1^0 = 0.3, 0.5, 0.7$. We present results for six values of the autoregressive parameter: $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$. Given the size distortions of the M test, all power comparisons are size-adjusted.

Tables 3.1 and 3.2 provide results pertaining to DGP-1. As expected, the power of all the tests decrease as α increases regardless of the location of the break. Power is also lower with serially correlated errors compared to the i.i.d. case, except when the errors contain a negative MA component. The tests are thus subject to a clear size-power trade-off in this latter case. The loss in power from introducing an autoregressive component in the errors is especially significant for the M test - power falls from 81% to 26% as ρ increases from zero to 0.5 when $\alpha = 0.5$ and $T = 150$. In comparison, the power performance of the proposed tests is much more robust to the extent of error serial correlation. Power also varies with the location of the break - power is high when the break occurs early in the sample ($\lambda_1^0 = 0.3$) and low when the break occurs relatively late ($\lambda_1^0 = 0.7$). This is due to the fact that the longer the $I(0)$ segment, the further away the series is from a pure unit root process. Relative to the proposed tests, however, the M test is much more sensitive to break location. A useful feature of the $W_1(1)$ test is that it rejects the null almost as frequently as the $\sup F_{1a}(1)$ test irrespective of the break location and the sample size. The proposed tests clearly outperform the M test in terms of power.

The results for DGP-2 are reported in Tables 4.1 and 4.2. Contrary to DGP-1, power is now higher when the break occurs late in the sample. The $\sup F_{1b}(1)$ test dominates the M test in most cases regardless of break location and sample size. The rejection probabilities of the M and $W_1(1)$ tests are broadly similar, except when the errors contain a pure negative MA component, in which case the M test rejects the null more often. Comparing the results for DGP-2 with those in DGP-1 also reveals that the cost in terms of power of not knowing the direction of shift is much higher when the true process involves an $I(0)$-$I(1)$ shift as opposed to an $I(1)$-$I(0)$ shift.

For DGP-3 and DGP-4, the results are presented in Tables 5.1 and 5.2 for $\lambda_1^0 = 0.5$. Again, the proposed tests generally outperform the M test for the break magnitudes and sample sizes considered. An exception is the $W_1(1)$ test which has lower power than both the $\sup F_{1b}(1)$ and M tests when the first regime is stationary. Finally, the rejection frequencies for DGP-5 reported in Table 6 indicate that, relative to DGP-1, the M test now has higher power while the proposed tests have lower power, though the latter still exhibits the highest power.

5.3 The Case With Two Breaks

With two breaks in persistence, we report results for three configurations for the locations of the breaks: $(\lambda_1^0, \lambda_2^0) = (0.3, 0.6), (0.3, 0.7), (0.4, 0.7)$. For the experiments in this section, we present results for the two breaks test, the $UDmax$ and $Wmax_1$ tests which do not require knowledge either of the direction or the number of breaks (except for an upper bound) and the M test. The DGPs considered are the following:

- DGP-6:

$$\begin{array}{lll} y_t = y_{t-1} + u_t & \text{if} & t \leq [T\lambda_1^0] \\ y_t = \alpha y_{t-1} + u_t & \text{if} & [T\lambda_1^0] + 1 \leq t \leq [T\lambda_2^0] \\ y_t = y_{t-1} + u_t & \text{if} & t \geq [T\lambda_2^0] + 1 \end{array}$$

- DGP-7:

$$\begin{array}{lll} y_t = \alpha y_{t-1} + u_t & \text{if} & t \leq [T\lambda_1^0] \\ y_t = y_{t-1} + u_t & \text{if} & [T\lambda_1^0] + 1 \leq t \leq [T\lambda_2^0] \\ y_t = \alpha y_{t-1} + u_t & \text{if} & t \geq [T\lambda_2^0] + 1 \end{array}$$

- DGP-8:

$$\begin{array}{rll} y_t = y_{t-1} + \pi_1 \Delta y_{t-1} + e_t & \text{if} & t \leq [T\lambda_1^0] \\ y_t = \alpha y_{t-1} + \pi_2 \Delta y_{t-1} + e_t & \text{if} & [T\lambda_1^0] + 1 \leq t \leq [T\lambda_2^0] \\ y_t = y_{t-1} + e_t & \text{if} & t \geq [T\lambda_2^0] + 1 \end{array}$$

- DGP-9:

$$\begin{array}{rll} y_t = \alpha y_{t-1} + \pi_1 \Delta y_{t-1} + e_t & \text{if} & t \leq [T\lambda_1^0] \\ y_t = y_{t-1} + \pi_2 \Delta y_{t-1} + e_t & \text{if} & [T\lambda_1^0] + 1 \leq t \leq [T\lambda_2^0] \\ y_t = \alpha y_{t-1} + e_t & \text{if} & t \geq [T\lambda_2^0] + 1 \end{array}$$

- DGP-10:

$$\begin{array}{rll} y_t = y_{t-1} + u_t & \text{if} & t \leq [T\lambda_1^0] \\ y_t - y_{[T\lambda_1^0]} = \alpha(y_{t-1} - y_{[T\lambda_1^0]}) + u_t & \text{if} & [T\lambda_1^0] + 1 \leq t \leq [T\lambda_2^0] \\ y_t = y_{t-1} + u_t & \text{if} & t \geq [T\lambda_2^0] + 1 \end{array}$$

First, consider the power of the various tests when the data are generated by DGP-6 and DGP-7. These results are presented in Tables 7.1-8.2. For DGP-6, the proposed tests clearly perform much better than the M test across location configurations and sample sizes. The $UDmax$ and $Wmax_1$ tests have power very close to that of the $\sup F_{1a}(2)$ test so that little is lost when the number of breaks is unknown. Note that the power of all tests is higher for $\lambda_1^0 = 0.3, \lambda_2^0 = 0.7$ compared to the other two location pairs. This is not unexpected since power should depend positively on the length of the $I(0)$ segment in the data. For DGP-7, our tests again dominate the M test except with pure negative MA errors, although the discrepancy in this latter case is not substantial. In accordance with the single break case, not knowing the number of breaks entails a non-negligible loss in power when the first regime is $I(0)$. The performance of the M test is again found to be quite sensitive to the location of the breaks for both DGP-6 and DGP-7.

The rejection frequencies for DGP-8 and DGP-9 are presented in Tables 9.1 and 9.2. For DGP-9, the rejection frequencies of the tests are close to those in the absence of regime-specific short-run dynamics. Surprisingly though, in the case of DGP-8, the proposed tests are more powerful relative to the case with no change in the short-run dynamics even though the tests are based against the alternative that these dynamics remain unchanged across regimes. Finally, the conclusions based on power results for DGP-10 reported in Table 10 are qualitatively similar to those discussed for DGP-5.

5.4 Summary and Practical Recommendations

In summary, the simulation results about the finite sample size of the tests reveal that our proposed tests are relatively better sized than that developed in Leybourne et al. (2007). The latter test have a substantial probability of over-rejection regardless of the degree of serial correlation in the errors. In most cases, the tests proposed are also shown to have a superior performance in terms of rejecting the null when the alternatives of interest drive the data generating process. Given the wide range of tests considered in this paper, some recommendations for applied work are in order. If the number of breaks is unknown but the direction of change is known under the alternative hypothesis, one can simply use the $UDmax$ test given that the test has power almost as high as that of the test of no change versus an alternative hypothesis that specifies the true number of changes. If the direction as well as the number of changes is unknown, one can apply the two $UDmax$ tests and examine which of them is significant. Since the test constructed against the alternative in which the initial regime has a unit root is not consistent against the alternative in which the initial regime is stationary, we can use this information to identify the initial regime. However, a rejection by both tests provides no conclusive evidence on the direction of change. In such a situation, we could rely on the $Wmax$ test but bearing in mind that the test has low power when the initial regime is stationary. Finally, it is important to note that the tests proposed should be applied after testing for a unit root using the whole sample. This is needed since our null hypothesis is that the process is $I(1)$ throughout the sample and ones needs to verify that it is not $I(0)$ throughout. If a rejection occurs, there is obviously no need to carry the change in persistence tests since standard unit root tests will have no power against processes which show changes in persistence so that upon a rejection one can safely conclude that the process is $I(0)$ throughout.

6 Conclusion

This paper has presented issues related to testing for multiple structural changes in the persistence of a univariate time series. In contrast to the existing literature which has primarily focused on sub-sample unit root tests and tests based on partial sums of residuals, we propose sup-Wald tests based on the difference between the sum of squared residuals under the null hypothesis of a unit root and that under the alternative hypothesis that the process displays changes in persistence over the sample. Our simulation experiments demonstrate that these tests have adequate finite sample properties. One important issue that we have not addressed is how to select the number of breaks. Indeed, we have assumed that the number of breaks is known a priori or less than some known upper bound. Bai and Perron (1998) propose a sequential strategy based on repeated application of the single break test in the context of stationary regression models. Such a strategy, however, does not seem to directly extend to our framework given that the process is stationary in only some regimes but has a unit root in others. Developing methods that would allow the consistent estimation of the number of breaks in this framework is an important avenue for future research. Finally, it is important to address the issue of the estimation of the break dates and develop method to form confidence intervals. These and other issues are the object of ongoing research.

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Appendix

As a matter of notation, throughout, we use the matrix norm $\|B\|_1 = \sup_{\|x\|\leq 1} \|Bx\|$, with $\|.\|$ the standard Euclidean norm. Note that $\|B\|_1$ equals the square root of the largest eigenvalue of $B'B$ and that $\|Bx\| \leq \|B\|_1 \|x\|$. Also, we use the usual norm $\|B\|^2 = tr(B'B)$, such that $\|B\|_1^2 \leq \|B\|^2$. Note that for any conformable matrices B_1 and B_2, we have $\|B_1 B_2\| \leq \|B_1\| \|B_2\|_1$. Next, we define $\bar{z}_j = (T_j - T_{j-1})^{-1} \sum_{t=T_{j-1}+1}^{T_j} z_t$ and $\bar{z}_{j,-1} = (T_j - T_{j-1})^{-1} \sum_{t=T_{j-1}+1}^{T_j} z_{t-1}$. Finally, we define the following regime-wise demeaned and detrended Brownian motions:

$$W^{(j)}(r) = W(r) - (\lambda_j - \lambda_{j-1})^{-1} \int_{\lambda_{j-1}}^{\lambda_j} W(r)dr$$

$$\widetilde{W}^{(j)}(r) = W^{(j)}(r) - \left[\frac{\int_{\lambda_{j-1}}^{\lambda_j} rW^{(j)}(r)dr}{\int_{\lambda_{j-1}}^{\lambda_j} \left\{ r - (\lambda_j - \lambda_{j-1})^{-1} \int_{\lambda_{j-1}}^{\lambda_j} rdr \right\}^2 dr} \right] \left[r - (\lambda_j - \lambda_{j-1})^{-1} \int_{\lambda_{j-1}}^{\lambda_j} rdr \right]$$

where $W(.)$ denotes a standard Brownian motion on $[0,1]$.

We first state a Lemma about the weak convergence of various sample moments whose proof is standard and thus omitted.

Lemma A.1 *If $\{w_t\}$ is generated as $w_t = w_{t-1} + v_t$, where v_t satisfies Assumption A2, the following weak convergence results hold (for $i = 1, ..., k+1$):*

a) $T^{-3/2} \sum_{t=1}^{[T\lambda_i]} w_t \Rightarrow \sigma \int_0^{\lambda_i} W(r)dr$,

b) $T^{-3/2} \sum_{t=1}^{[T\lambda_i]} w_t^2 \Rightarrow \sigma^2 \int_0^{\lambda_i} W(r)^2 dr$

c) $T^{-1} \sum_{t=1}^{[T\lambda_i]} w_{t-1} u_t \Rightarrow \sigma^2 \int_0^{\lambda_i} W(r)dW(r)$

Proof of Theorem 1: We shall prove the theorem for models 1a and 2a. The proofs for the other models are similar and hence omitted.

Model 1a: We have

$$y_t = c_i + \alpha_i y_{t-1} + u_t, \quad t = T_{i-1}+1, ..., T_i$$

for $i = 1, ..., k+1$ with $\alpha_i = 1,\ c_i = 0$ in odd regimes and $|\alpha_i| < 1,\ c_i$ unrestricted in even regimes. Under the null hypothesis of a unit root throughout the sample, the sum of squared residuals is

$$SSR_0 = \sum_{t=1}^{T} (y_t - y_{t-1})^2 = \sum_{t=1}^{T} u_t^2$$

If k is even, the sum of squares residuals under the alternative hypothesis is

$$SSR_{1a,k} = \sum_{i=1}^{k/2} \left[\sum_{t=T_{2i-1}+1}^{T_{2i}} \{y_t - \bar{y}_{2i} - \hat{\alpha}_{2i}(y_{t-1} - \bar{y}_{2i,-1}\}^2 \right] + \sum_{i=0}^{k/2} \sum_{t=T_{2i}+1}^{T_{2i+1}} u_t^2 \qquad \text{(A.1)}$$

where, for $i = 1, ..., k/2$,

$$\hat{\alpha}_{2i} = \frac{\sum_{t=T_{2i-1}+1}^{T_{2i}} (y_t - \bar{y}_{2i})(y_{t-1} - \bar{y}_{2i,-1})}{\sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1})^2}$$

Note that, under the null, $y_t = y_{t-1} + u_t$ which implies $\bar{y}_{2i} = \bar{y}_{2i,-1} + \bar{u}_{2i}$. Substituting in the expression for $\hat{\alpha}_{2i}$ and using Lemma A.1, we have

$$T(\hat{\alpha}_{2i} - 1) = \frac{T^{-1} \sum_{t=T_{2i-1}+1}^{T_{2i}} (y_t - \bar{y}_{2i}) u_t}{T^{-2} \sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1})^2} \Rightarrow \frac{\int_{\lambda_{2i-1}}^{\lambda_{2i}} W^{(2i)}(r) dW(r)}{\int_{\lambda_{2i-1}}^{\lambda_{2i}} \left[W^{(2i)}(r)\right]^2 dr}$$

From $(A.1)$, we thus have, under the null hypothesis,

$$\begin{aligned}
& SSR_{1a,k} \\
= & \sum_{i=1}^{k/2} \left[\frac{-\left\{ \sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1}) u_t \right\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1})^2} + \sum_{t=T_{2i-1}+1}^{T_{2i}} (u_t - \bar{u}_{2i})^2 \right] + \sum_{i=0}^{k/2} \sum_{t=T_{2i}+1}^{T_{2i+1}} u_t^2 \\
= & \sum_{i=1}^{k/2} \left[\frac{-\left\{ \sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1}) u_t \right\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1})^2} - \frac{T}{T_{2i} - T_{2i-1}} \left\{ T^{-1/2} \sum_{t=T_{2i-1}+1}^{T_{2i}} u_t \right\}^2 \right] + \sum_{t=1}^{T} u_t^2
\end{aligned}$$

so that

$$\begin{aligned}
& SSR_0 - SSR_{1a,k} \\
= & \sum_{i=1}^{k/2} \left[\frac{\left\{ \sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1}) u_t \right\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1})^2} + \frac{T}{T_{2i} - T_{2i-1}} \left\{ T^{-1/2} \sum_{t=T_{2i-1}+1}^{T_{2i}} u_t \right\}^2 \right] \\
\Rightarrow & \sigma^2 \sum_{i=1}^{k/2} \left[\frac{\left\{ \int_{\lambda_{2i-1}}^{\lambda_{2i}} W^{(2i)}(r) dW(r) \right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}} [W^{(2i)}(r)]^2 dr} + \frac{1}{\lambda_{2i} - \lambda_{2i-1}} \left\{ W(\lambda_{2i}) - W(\lambda_{2i-1}) \right\}^2 \right]
\end{aligned}$$

It is easy to show that

$$T^{-1} SSR_{1a,k} = T^{-1} \sum_{t=1}^{T} u_t^2 + o_p(1) \xrightarrow{p} \sigma^2$$

so that

$$kF_{1a}(\lambda,k) \Rightarrow \sum_{i=1}^{k/2}\left[\frac{\left\{\int_{\lambda_{2i-1}}^{\lambda_{2i}} W^{(2i)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}[W^{(2i)}(r)]^2dr} + \frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i})-W(\lambda_{2i-1})\right\}^2\right]$$

If k is odd,

$$SSR_{1a,k} = \sum_{i=0}^{(k-1)/2}\sum_{t=T_{2i}}^{T_{2i+1}} u_t^2 + \sum_{i=1}^{(k+1)/2}\left[\sum_{t=T_{2i-1}+1}^{T_{2i}}\left\{y_t-\bar{y}_{2i}-\hat{\alpha}_{2i}(y_{t-1}-\bar{y}_{2i,-1}\right\}^2\right]$$

and similar derivations show that

$$(k+1)F_{1a}(\lambda,k) \Rightarrow \sum_{i=1}^{(k+1)/2}\left[\frac{\left\{\int_{\lambda_{2i-1}}^{\lambda_{2i}} W^{(2i)}(r)dW(r)\right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}[W^{(2i)}(r)]^2dr} + \frac{1}{\lambda_{2i}-\lambda_{2i-1}}\left\{W(\lambda_{2i})-W(\lambda_{2i-1})\right\}^2\right]$$

Model 2a: We have

$$y_t = c_i + b_i t + \alpha_i y_{t-1} + u_t, \quad t = T_{i-1}+1, ..., T_i,$$

with $\alpha_i = 1,\ b_i = 0,\ c_i$ unrestricted in odd regimes and $|\alpha_i| < 1,\ b_i,\ c_i$ unrestricted in even regimes. Under the null, $y_t = c + y_{t-1} + u_t$. For this model, we have

$$SSR_0^* = \sum_{t=1}^{T}\left[y_t - y_{t-1} - T^{-1}\sum_{t=1}^{T}(y_t-y_{t-1})\right]^2 = \sum_{t=1}^{T}(u_t-\bar{u})^2$$

Again, consider first the case with k even. For $t \in [T_{2i-1}+1, T_{2i}]$, define

$$\begin{aligned}
\widetilde{y}_t &= y_t - \bar{y}_{2i} - \frac{\sum_{t=T_{2i-1}+1}^{T_{2i}}(y_t-\bar{y}_{2i})(t-\bar{t}_{2i})}{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2}(t-\bar{t}_{2i}) \\
\widetilde{y}_{t-1} &= y_{t-1} - \bar{y}_{2i,-1} - \frac{\sum_{t=T_{2i-1}+1}^{T_{2i}}(y_{t-1}-\bar{y}_{2i,-1})(t-\bar{t}_{2i})}{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2}(t-\bar{t}_{2i})
\end{aligned}$$

Then, under the null hypothesis, we can write

$$\widetilde{y}_t = \widetilde{y}_{t-1} + u_t - \bar{u}_{2i} - \frac{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})u_t}{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2}(t-\bar{t}_{2i}) \tag{A.2}$$

We have

$$\begin{aligned}
SSR_{2a,k} &= \sum_{i=1}^{k/2}\left[\sum_{t=T_{2i-1}+1}^{T_{2i}}\left\{\widetilde{y}_t - \widetilde{\alpha}_{2i}\widetilde{y}_{t-1}\right\}^2\right] \\
&+ \sum_{i=0}^{k/2}\left[\sum_{t=T_{2i}+1}^{T_{2i+1}}\left\{y_t - y_{t-1} - \frac{1}{T_{2i+1}-T_{2i}}\sum_{t=T_{2i}+1}^{T_{2i+1}}(y_t-y_{t-1})\right\}^2\right]
\end{aligned} \tag{A.3}$$

where

$$\widetilde{\alpha}_{2i} = \frac{\sum_{t=T_{2i-1}+1}^{T_{2i}} \widetilde{y}_t \widetilde{y}_{t-1}}{\sum_{t=T_{2i-1}+1}^{T_{2i}} \widetilde{y}_{t-1}^2} \tag{A.4}$$

Using $(A.2)$ and $(A.4)$, we can express $(A.3)$ as

$$\begin{aligned} SSR_{2a,k} &= \sum_{i=1}^{k/2} \left[\frac{-\{\sum_{t=T_{2i-1}+1}^{T_{2i}} \widetilde{y}_{t-1} u_t\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}} \widetilde{y}_{t-1}^2} + \sum_{t=T_{2i-1}+1}^{T_{2i}} (u_t - \bar{u}_{2i})^2 - \frac{\{\sum_{t=T_{2i-1}+1}^{T_{2i}} (t - \bar{t}_{2i}) u_t\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}} (t - \bar{t}_{2i})^2} \right] \\ &\quad + \sum_{i=0}^{k/2} \sum_{t=T_{2i}+1}^{T_{2i+1}} (u_t - \bar{u}_{2i+1})^2 \end{aligned}$$

We thus get

$$\begin{aligned} SSR_0^* - SSR_{2a,k} &= -\left(T^{-1/2} \sum_{t=1}^{T} u_t\right)^2 + \sum_{i=0}^{k/2} \left[\frac{T}{T_{2i+1} - T_{2i}} \left(T^{-1/2} \sum_{t=T_{2i}+1}^{T_{2i+1}} u_t\right)^2 \right] \\ &\quad + \sum_{i=1}^{k/2} \left[\begin{array}{c} \frac{\{\sum_{t=T_{2i-1}+1}^{T_{2i}} \widetilde{y}_{t-1} u_t\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}} \widetilde{y}_{t-1}^2} + \frac{T}{T_{2i} - T_{2i-1}} \left\{T^{-1/2} \sum_{t=T_{2i-1}+1}^{T_{2i}} u_t\right\}^2 \\ + \frac{\{\sum_{t=T_{2i-1}+1}^{T_{2i}} (t - \bar{t}_{2i}) u_t\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}} (t - \bar{t}_{2i})^2} \end{array} \right] \end{aligned}$$

which yields

$$\begin{aligned} 2kF_{2a}(\lambda, k) &\Rightarrow -\{W(1)\}^2 + \sum_{i=0}^{k/2} \left[\frac{1}{\lambda_{2i+1} - \lambda_{2i}} \{W(\lambda_{2i+1}) - W(\lambda_{2i})\}^2 \right] \\ &\quad + \sum_{i=1}^{k/2} \left[\begin{array}{c} \frac{\left\{\int_{\lambda_{2i-1}}^{\lambda_{2i}} \widetilde{W}^{(2i)}(r) dW(r)\right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}} [\widetilde{W}^{(2i)}(r)]^2 dr} + \frac{1}{\lambda_{2i} - \lambda_{2i-1}} \{W(\lambda_{2i}) - W(\lambda_{2i-1})\}^2 \\ + \frac{\left[\int_{\lambda_{2i-1}}^{\lambda_{2i}} \{r - (\lambda_{2i} - \lambda_{2i-1})^{-1} \int_{\lambda_{2i-1}}^{\lambda_{2i}} r dr\} dW(r)\right]^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}} \{r - (\lambda_{2i} - \lambda_{2i-1})^{-1} \int_{\lambda_{2i-1}}^{\lambda_{2i}} r dr\}^2 dr} \end{array} \right] \end{aligned}$$

If k is odd,

$$\begin{aligned} SSR_{2a,k} &= \sum_{i=0}^{(k-1)/2} \sum_{t=T_{2i}+1}^{T_{2i+1}} \{y_t - y_{t-1} - \frac{1}{T_{2i+1} - T_{2i}} \sum_{t=T_{2i}+1}^{T_{2i+1}} (y_t - y_{t-1})\}^2 \\ &\quad + \sum_{i=1}^{(k+1)/2} \left[\sum_{t=T_{2i-1}+1}^{T_{2i}} \{\widetilde{y}_t - \widetilde{\alpha}_{2i} \widetilde{y}_{t-1}\}^2 \right] \end{aligned}$$

and similar derivations yield the result stated in Theorem 1.

Given these limits, the results of Theorem 1 follow from an application of the Continuous Mapping Theorem. ■

We will prove Theorem 2 for model 1a when k is even. The proof is similar for the other cases. The autoregression in the i-th regime ($i = 1, ..., k/2$) is

$$\Delta y_t = c_{2i} + (\alpha_{2i} - 1)y_{t-1} + \sum_{j=1}^{l_T} \pi_j \Delta y_{t-j} + v_t^* \tag{A.5}$$

with $v_t^* = e_t + v_t$, and $e_t = \sum_{j>l_T} \pi_j \Delta y_{t-j}$. Let $\eta_t' = (\Delta y_{t-1}, ..., \Delta y_{t-l_T})$, $\eta = (\eta_1, ..., \eta_T)'$, $\Pi = (\pi_1, ..., \pi_{l_T})'$, $V^* = (v_1^*, ..., v_T^*)' = V + \mathcal{E}$ with $V = (v_1, ..., v_T)'$ and $\mathcal{E} = (e_1, ...e_T)'$. We can write $(A.5)$ as

$$\Delta y_t = c_i + (\alpha_i - 1)y_{t-1} + \eta_t'\Pi + v_t^*$$

with $\alpha_i = 1$, $c_i = 0$ in odd regimes and $|\alpha_i| < 1$, c_i unrestricted in even regimes. For $j = 1, ..., k+1$, we denote $\Delta Y_j = (\Delta y_{T_{j-1}+1}, ..., \Delta y_{T_j})'$, $\eta_j^* = (\eta_{T_{j-1}+1}, ..., \eta_{T_j})'$, $\mathcal{E}_j = (e_{T_{j-1}+1}, ..., e_{T_j})'$, $V_j = (v_{T_{j-1}+1}, ..., v_{T_j})'$ and $V_j^* = (v^*_{T_{j-1}+1}, ..., v^*_{T_j})'$. For $i = 1, ..., k/2$, let $\hat{\gamma}_{2i} = (\hat{c}_{2i}, \hat{\alpha}_{2i} - 1)'$ and $Z_{2i} = (z_{T_{2i-1}+1}, ..., z_{T_{2i}})'$ where $z_t = (1, y_{t-1})'$ for $t = T_{2i-1}+1, ..., T_{2i}$. Define the (2×2) diagonal matrix $D_T = diag(T^{-1/2}, T^{-1})$.

The proof of Theorem 2 is based on the following Lemma.

Lemma A.2 *Assume y_t is generated as $y_t = y_{t-1} + u_t$. Under A1-A3, we have (a) $||\,(\eta'\eta)^{-1}\,||_1 = O_p(T^{-1})$, (b) (i) $||D_T Z_{2i}'\eta_{2i}^*|| = O_p(l_T^{1/2})$ and (ii) $||D_T Z_{2i}'\mathcal{E}_{2i}|| = o_p(l_T^{-1})$, for $i = 1, ..., k/2$, (c) $||\eta' V|| = O_p(T^{1/2} l_T^{1/2})$, (d) $||\eta'\mathcal{E}|| = o_p(T l_T^{-1/2})$, (e) $||\mathcal{E}'\mathcal{E}|| = o_p(T)$, (f) $||\mathcal{E}'V|| = o_p(T)$, (g) $||\eta' V^*|| = o_p(T l_T^{-1/2})$, (h) $\left\| \left[\eta'\eta - \sum_{i=1}^{k/2} \{\eta_{2i}^{*\prime} Z_{2i}(Z_{2i}'Z_{2i})^{-1} Z_{2i}'\eta_{2i}^*\}\right]^{-1} \right\|_1 = O_p(T^{-1})$*

Proof of Lemma A.2: (a) Let $\Gamma_l^* = (\Gamma_{i-j})_{i,j=1}^{l_T}$, where $\Gamma_h = E(u_t u_{t-h})$. From Berk (1974, Lemma 3), it follows that $E||\,(T^{-1}\eta'\eta)^{-1} - (\Gamma_l^*)^{-1}||_1^2 \leq C_1 T^{-1} l_T^2$ for some constant C_1. Hence, $||\,(T^{-1}\eta'\eta)^{-1} - (\Gamma_l^*)^{-1}||_1 = O_p(T^{-1/2} l_T)$. Since $\|(\Gamma_l^*)^{-1}\|_1 = O(1)$ uniformly in l_T for sequences such that $T^{-1/2} l_T \to 0$, we have

$$\left| \left\| (T^{-1}\eta'\eta)^{-1} \right\|_1 - \left\|(\Gamma_l^*)^{-1}\right\|_1 \right| \leq \left\| (T^{-1}\eta'\eta)^{-1} - (\Gamma_l^*)^{-1} \right\|_1 = o_p(1)$$

and the result follows.

(b) For (i), the result follows since each element of $D_T Z_{2i}'\eta_{2i}^*$ is $O_p(1)$ and the number of elements is of order $O(l_T)$. For (ii), The result follows from Lemma A.2(a) of Lütkepohl and Saikkonen (1999).

(c) The elements of $T^{-1/2}\eta' V$ are each $O_p(1)$ (since each element of η_t and v_t are uncorrelated), and the result follows since the number of elements is of order $O(l_T)$.

(d) We have

$$
\begin{aligned}
E\left\|T^{-1}\eta'\mathcal{E}\right\| &\leq T^{-1}\sum_{t=1}^{T}E(\|e_t\eta_t\|)\leq\{E(\|\eta_t\|^2)E(e_t^2)\}^{1/2}=C_2 l_T^{1/2}\{E(\sum_{j>l_T}\Delta y'_{t-j}\Pi_j)^2\}^{1/2}\\
&\leq C_2 l_T^{1/2}\{\sum_{i>l_T}\sum_{j>l_T}|\Gamma_{i-j}|\,|\Pi_i|\,|\Pi_j|\}^{1/2}\leq C_3 l_T^{1/2}\sum_{j>l_T}|\Pi_j|=o(l_T^{-1/2})
\end{aligned}
$$

using the fact that $|\Gamma_{i-j}|$ is uniformly bounded by the stationarity of u_t.

(e) We have

$$
\begin{aligned}
E\left\|T^{-1}\mathcal{E}'\mathcal{E}\right\| &= T^{-1}\sum_{t=1}^{T}E(e_t^2)=T^{-1}\sum_{t=1}^{T}\sum_{i>l_T}\sum_{a>l_T}\Pi_i E(\Delta y_{t-i}\Delta y'_{t-a})\Pi_a\\
&\leq T^{-1}\sum_{i>l_T}\sum_{a>l_T}\sum_{t=1}^{T}|\Pi_i|\,|\Gamma_{a-i}|\,|\Pi_a|\leq o(l_T^{-2})=o(1)
\end{aligned}
$$

where we again use the fact that $|\Gamma_j|$ is bounded uniformly in j.

(f) We have $T^{-1}\sum_{t=1}^{T}v_te_t=T^{-1}\sum_{i>l_T}\Pi_i'\sum_{t=1}^{T}\Delta y_{t-i}v_t$, so that

$$
||T^{-1}\sum_{t=1}^{T}v_te_t||\leq T^{-1}\sum_{i>l_T}\|\Pi_i\|\,||\sum_{t=1}^{T}\Delta y_{t-i}v_t||=o_p(l_T^{-1}T^{-1/2})=o_p(1)
$$

where we used the fact that $T^{-1/2}\sum_{t=1}^{T}\Delta y_{t-i}v_t=O_p(1)$.

(g) Since $V^*=V+\mathcal{E}$, $||\eta'V^*||\leq||\eta'V||+||\eta'\mathcal{E}||=O_p(T^{1/2}l_T^{1/2})+o_p(Tl_T^{-1/2})=o_p(Tl_T^{-1/2})$.

(h) Let

$$
q=\left\|\left[T^{-1}\eta'\eta-T^{-1}\sum_{i=1}^{k/2}\left\{\eta_{2i}^{*\prime}Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'\eta_{2i}^*\right\}\right]^{-1}-(\Gamma_l^*)^{-1}\right\|_1
$$

and

$$
Q=||T^{-1}\eta'\eta-T^{-1}\sum_{i=1}^{k/2}\left\{\eta_{2i}^{*\prime}Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'\eta_{2i}^*\right\}-\Gamma_l^*||_1.
$$

Then we have

$$
q\leq\left\{q+\left\|(\Gamma_l^*)^{-1}\right\|_1\right\}Q\left\|(\Gamma_l^*)^{-1}\right\|_1
$$

or,

$$
q\leq\frac{\|(\Gamma_l^*)^{-1}\|_1^2\,Q}{1-Q\left\|(\Gamma_l^*)^{-1}\right\|_1} \tag{A.6}
$$

Also,

$$
\begin{aligned}
Q &\leq \left\|T^{-1}\eta'\eta - \Gamma_l^*\right\|_1 + \left\|T^{-1}\sum_{i=1}^{k/2}\left\{\eta_{2i}^{*\prime}Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'\eta_{2i}^*\right\}\right\| \\
&= \left\|T^{-1}\eta'\eta - \Gamma_l^*\right\|_1 + T^{-1}\sum_{i=1}^{k/2}\left\{\left\|\eta_{2i}^{*\prime}Z_{2i}D_T\right\|\left\|(D_TZ_{2i}'Z_{2i}D_T)^{-1}\right\|\left\|D_TZ_{2i}'\eta_{2i}^*\right\|\right\} \\
&= O_p(l_T/T^{1/2}) + T^{-1}O_p(l_T^{1/2})O_p(1)O_p(l_T^{1/2}) = O_p(l_T/T^{1/2})
\end{aligned}
$$

Since $\|(\Gamma_l^*)^{-1}\|_1 = O_p(1)$, from $(A.6)$, we get $q = O_p(l_T/T^{1/2})$. We thus have

$$
\left|\left\|\left[T^{-1}\eta'\eta - T^{-1}\sum_{i=1}^{k/2}\left\{\eta_{2i}^{*\prime}Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'\eta_{2i}^*\right\}\right]^{-1}\right\|_1 - \left\|(\Gamma_l^*)^{-1}\right\|_1\right| = O_p(l_T/T^{1/2}) = o_p(1)
$$

so that

$$
\left\|\left[T^{-1}\eta'\eta - T^{-1}\sum_{i=1}^{k/2}\left\{\eta_{2i}^{*\prime}Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'\eta_{2i}^*\right\}\right]^{-1}\right\|_1 = O_p(1)
$$

and the result follows. ■

Proof of Theorem 2: For $i = 1, ..., k+1$, we denote the vector of residuals in the j-th regime under the null and alternative hypotheses by $\widetilde{V}_i^*$ and $\hat{V}_i^*$ respectively. Then we have

$$
\begin{aligned}
\widetilde{V}_i^* &= \Delta Y_i - \eta_i^*\widetilde{\Pi}, && \text{for } i = 1, ..., k+1 \\
\hat{V}_{2i}^* &= \Delta Y_{2i} - \eta_{2i}^*\hat{\Pi} - Z_{2i}\hat{\gamma}_{2i}, && \text{for } i = 1, ..., k/2 \\
\hat{V}_{2i+1}^* &= \Delta Y_{2i} - \eta_{2i}^*\hat{\Pi}, && \text{for } i = 0, ..., k/2
\end{aligned}
\tag{A.7}
$$

where

$$
\widetilde{\Pi} - \Pi = (\eta'\eta)^{-1}\eta'V^* \tag{A.8}
$$

under H_0. Also, $\hat{\Pi}$ and $\hat{\gamma}_{2i}$ satisfy the first order conditions

$$
Z_{2i}'\hat{V}_{2i}^* = 0, \text{ for } i = 1, ..., k/2 \tag{A.9}
$$

$$
\sum_{i=1}^{k/2}\eta_{2i}^*\hat{V}_{2i}^* + \sum_{i=0}^{k/2}\eta_{2i+1}^*\hat{V}_{2i+1}^* = 0 \tag{A.10}
$$

Under H_0, from $(A.10)$, we have

$$
\hat{\Pi} - \Pi = (\eta'\eta)^{-1}(\eta'V^* - \sum_{i=1}^{k/2}\eta_{2i}^{*\prime}Z_{2i}\hat{\gamma}_{2i}) \tag{A.11}
$$

Next, from $(A.9)$, for $i = 1, ..., k/2$,

$$D_T^{-1}\hat{\delta}_{2i} = (D_T Z'_{2i} Z_{2i} D_T)^{-1}\left[D_T Z'_{2i}\eta^*_{2i}(\Pi - \hat{\Pi}) + D_T Z'_{2i}\mathcal{E}_{2i} + D_T Z'_{2i} V_{2i}\right] \tag{A.12}$$

Solving for $(\hat{\Pi} - \Pi)$ from $(A.12)$ and $(A.11)$, we get

$$\hat{\Pi} - \Pi = \left[\eta'\eta - \sum_{i=1}^{k/2}\left\{\eta^{*\prime}_{2i} Z_{2i}(Z'_{2i}Z_{2i})^{-1} Z'_{2i}\eta^*_{2i}\right\}\right]^{-1}\left[\eta' V^* - \sum_{i=1}^{k/2}\left\{\eta^{*\prime}_{2i} Z_{2i}(Z'_{2i}Z_{2i})^{-1} Z'_{2i} V^*_{2i}\right\}\right] \tag{A.13}$$

Using parts (b), (g) and (h) of Lemma A.2, we get $||\hat{\Pi} - \Pi|| = o_p(l_T^{-1/2})$. Then we have, using Lemma A.2(b),

$$\begin{aligned}\left\|(D_T Z'_{2i} Z_{2i} D_T)^{-1} D_T Z'_{2i}\eta^*_{2i}(\Pi - \hat{\Pi})\right\| &\leq \left\|(D_T Z'_{2i} Z_{2i} D_T)^{-1}\right\| \|D_T Z'_{2i}\eta^*_{2i}\| \left\|(\Pi - \hat{\Pi})\right\| \\ &= O_p(1).O_p(l_T^{1/2}) o_p(l_T^{-1/2}) = o_p(1)\end{aligned}$$

Also, $\|D_T Z'_{2i}\mathcal{E}_{2i}\| = o_p(l_T^{-1})$. Using this in $(A.12)$, we have

$$D_T^{-1}\hat{\gamma}_{2i} = (D_T Z'_{2i} Z_{2i} D_T)^{-1} D_T Z'_{2i} V_{2i} + o_p(1) \tag{A.14}$$

Further, from $(A.8)$ and $(A.13)$, we get

$$\hat{\Pi} - \widetilde{\Pi} = -(\eta'\eta)^{-1}\sum_{i=1}^{k/2}\{\eta^{*\prime}_{2i} Z_{2i}\hat{\gamma}_{2i}\} \tag{A.15}$$

so that

$$\begin{aligned}\left\|\hat{\Pi} - \widetilde{\Pi}\right\| &\leq \left\|(\eta'\eta)^{-1}\right\|_1 \left\|\sum_{i=1}^{k/2}\left\{(\eta^{*\prime}_{2i} Z_{2i} D_T)(D_T^{-1}\hat{\gamma}_{2i})\right\}\right\| \\ &\leq \left\|(\eta'\eta)^{-1}\right\|_1 \sum_{i=1}^{k/2}\|\eta^{*\prime}_{2i} Z_{2i} D_T\| \left\|D_T^{-1}\hat{\gamma}_{2i}\right\| = O_p(l_T^{1/2} T^{-1})\end{aligned} \tag{A.16}$$

We can write, from $(A.7)$, for $i = 1, ..., k/2$,

$$\widetilde{v}^*_{2i} = \hat{v}^*_{2i} + Z_{2i}\hat{\gamma}_{2i} + \eta^*_{2i}(\hat{\Pi} - \widetilde{\Pi})$$

and for $i = 0, ..., k/2$,

$$\widetilde{v}^*_{2i+1} = \hat{v}^*_{2i+1} + \eta^*_{2i+1}(\hat{\Pi} - \widetilde{\Pi})$$

Thus the numerator of the F statistic can be written as

$$\begin{aligned} SSR_0 - SSR_{1a,k} &= \sum_{i=1}^{k/2}\{\widetilde{v}^{*\prime}_{2i}\widetilde{v}^*_{2i} - \hat{v}^{*\prime}_{2i}\hat{v}^*_{2i}\} + \sum_{i=0}^{k/2}\{\widetilde{v}^{*\prime}_{2i+1}\widetilde{v}^*_{2i+1} - \hat{v}^{*\prime}_{2i+1}\hat{v}^*_{2i+1}\} \\ &= \sum_{i=1}^{k/2}(D_T^{-1}\hat{\gamma}_{2i})(D_T Z'_{2i} Z_{2i} D_T) D_T^{-1}\hat{\gamma}_{2i} + (\hat{\Pi} - \widetilde{\Pi})'\sum_{i=1}^{k/2}(\eta^{*\prime}_{2i} Z_{2i} D_T)(D_T^{-1}\hat{\gamma}_{2i})\end{aligned} \tag{A.17}$$

Next,

$$\left\| (\hat{\Pi} - \widetilde{\Pi})' \sum_{i=1}^{k/2} (\eta_{2i}^{*\prime} Z_{2i} D_T)(D_T^{-1} \hat{\gamma}_{2i}) \right\| \leq \left\| \hat{\Pi} - \widetilde{\Pi} \right\| \sum_{i=1}^{k/2} \|(\eta_{2i}^{*\prime} Z_{2i} D_T)\| \left\|(D_T^{-1} \hat{\gamma}_{2i})\right\|$$

$$= O_p(l_T^{1/2} T^{-1}).O_p(l_T^{1/2}).O_p(1) = O_p(l_T T^{-1}) = o_p(1)$$

Then, using $(A.14)$ in $(A.17)$, we have

$$SSR_0 - SSR_{1a,k} = \sum_{i=1}^{k/2} \left\{ V_{2i}' Z_{2i} (D_T Z_{2i}' Z_{2i} D_T)^{-1} D_T Z_{2i}' V_{2i} \right\} + o_p(1) \qquad \text{(A.18)}$$

Under H_0, we have the Beveridge-Nelson decomposition,

$$y_t = d(1) w_t + \bar{u}_0 - \bar{u}_t$$

where $w_t = \sum_{j=1}^{t} v_j$, $\bar{u}_t = \sum_{s=0}^{\infty} \bar{d}_s v_{t-s}$, $\bar{d}_s = \sum_{i=s+1}^{\infty} d_i$. Note that $(\bar{u}_t)$ is stochastically of smaller order of magnitude than (w_t). Then for $r \in (0,1]$, we have $T^{-2} \sum_{t=1}^{[Tr]} y_t^2 = d(1)^2 T^{-2} \sum_{t=1}^{[Tr]} w_t^2 + o_p(1)$ and $T^{-1} \sum_{t=1}^{[T\lambda_i]} y_{t-1} v_t = d(1) T^{-1} \sum_{t=1}^{[Tr]} w_{t-1} v_t + o_p(1)$. Using these results in $(A.18)$, we get

$$SSR_0 - SSR_{1a,k} \Rightarrow \sigma^2 \sum_{i=1}^{k/2} \left[\frac{\left\{ \int_{\lambda_{2i-1}}^{\lambda_{2i}} W^{(2i)}(r) dW(r) \right\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}} [W^{(2i)}(r)]^2 dr} + \frac{1}{\lambda_{2i} - \lambda_{2i-1}} \left\{ W(\lambda_{2i}) - W(\lambda_{2i-1}) \right\}^2 \right]$$

Using the fact that $T^{-1} SSR_{1a,k} \xrightarrow{p} \sigma^2$, the result follows. ■

Proof of Theorem 3: We will prove the result for Model 1a and k even. To show that the test is consistent, we will show that for $\lambda = (\lambda_1^0, ..., \lambda_k^0)$, the true break fractions, the test diverges at rate T. To see this, first note that, under the alternative hypothesis, $(A.17)$ still holds. Next, we have

$$\left\| \hat{\Pi} - \widetilde{\Pi} \right\| \leq \left\| (\eta' \eta)^{-1} \right\| \sum_{i=1}^{k/2} \|\eta_{2i}^{*\prime} Z_{2i}\| \, \|\hat{\gamma}_{2i}\|$$

Then, using the fact that $\hat{\gamma}_{2i} \xrightarrow{p} \gamma_{2i}^0$ where $\gamma_{2i}^0 = (c_{2i}^0, \alpha_{2i}^0 - 1)'$ denotes the true value of γ_{2i}, as well as the fact that $\|\eta_{2i}^{*\prime} Z_{2i}\| = O_p(T)$ and $\|(\eta'\eta)^{-1}\| = O_p(T^{-1})$, we get $||\hat{\Pi} - \widetilde{\Pi}|| = O_p(1)$ so that the second term of $(A.17)$ is $O_p(T)$. The first term is also $O_p(T)$ since $Z_{2i}' Z_{2i} = O_p(T)$. Given that $T^{-1} SSR_{1a,k} \xrightarrow{p} \sigma^2$, the result follows. ■

Table 1: Asymptotic Critical Values

(A) Non-trending Case

Sig. Level	$\sup F_{1a}(\lambda,k)$ Number of Breaks, k: 1	2	3	4	5	$UDmax_{1a}$	$\sup F_{1b}(\lambda,k)$ Number of Breaks, k: 1	2	3	4	5	$UDmax_{1b}$	$W_1(k)$ Number of Breaks, k: 1	2	3	4	5	$Wmax_1$
10%	7.94	9.47	7.08	7.04	5.11	9.84	5.41	5.64	6.05	5.33	4.84	6.67	8.08	9.51	7.28	7.10	5.40	9.86
5%	8.88	10.62	7.73	7.67	5.56	10.87	6.39	6.33	6.68	5.84	5.29	7.36	8.99	10.62	7.91	7.71	5.79	10.90
2.5%	9.93	11.64	8.33	8.30	5.95	11.85	7.28	6.84	7.35	6.31	5.70	7.99	10.00	11.64	8.49	8.32	6.21	11.95
1%	11.11	12.72	9.19	9.05	6.46	13.00	8.28	7.42	8.04	6.87	6.17	8.64	11.21	12.72	9.44	9.05	6.63	13.02

(B) Trending Case

Sig. Level	$\sup F_{2a}(\lambda,k)$ Number of Breaks, k: 1	2	3	4	5	$UDmax_{2a}$	$\sup F_{2b}(\lambda,k)$ Number of Breaks, k: 1	2	3	4	5	$UDmax_{2b}$	$W_2(k)$ Number of Breaks, k: 1	2	3	4	5	$Wmax_2$
10%	7.07	6.90	5.78	5.36	4.27	7.61	5.67	5.50	5.24	4.82	4.12	6.17	7.28	7.01	5.96	5.48	4.46	7.71
5%	7.84	7.57	6.18	5.77	4.57	8.27	6.52	6.02	5.67	5.17	4.39	6.78	7.98	7.60	6.36	5.86	4.74	8.43
2.5%	8.49	8.20	6.56	6.14	4.80	9.07	7.12	6.43	6.08	5.47	4.69	7.27	8.75	8.22	6.77	6.18	4.98	9.18
1%	9.64	9.15	7.23	6.59	5.14	10.01	8.07	7.00	6.59	5.82	4.97	8.17	9.73	9.18	7.30	6.63	5.34	10.07

Table 2.1: Empirical Size (DGP-0, $\pi_1 = \pi_2 = 0$, Nominal Size = 5%)

	$\rho = \theta = 0$		$\rho = 0.3,\ \theta = 0$		$\rho = 0.5,\ \theta = 0$		$\rho = 0,\ \theta = 0.5$		$\rho = 0,\ \theta = -0.5$		$\rho = 0.3,\ \theta = 0.5$		$\rho = 0.3,\ \theta = -0.5$	
	$T = 150$	$T = 240$	$T = 150$	$T = 240$	$T = 150$	$T = 240$	$T = 150$	$T = 240$	$T = 150$	$T = 240$	$T = 150$	$T = 240$	$T = 150$	$T = 240$
sup $F_{1a}(1)$	.05	.06	.07	.06	.05	.06	.06	.07	.10	.07	.05	.07	.11	.10
sup $F_{1a}(2)$	.04	.05	.04	.06	.03	.05	.05	.07	.07	.06	.06	.06	.10	.12
$UDmax_{1a}$	.05	.05	.06	.05	.04	.05	.07	.07	.14	.08	.07	.07	.12	.16
sup $F_{1b}(1)$	.07	.05	.07	.06	.07	.06	.10	.07	.11	.11	.09	.07	.16	.15
sup $F_{1b}(2)$	.06	.05	.09	.06	.08	.05	.11	.07	.19	.12	.10	.08	.18	.18
$UDmax_{1b}$	.07	.07	.10	.08	.07	.06	.13	.08	.31	.15	.11	.09	.23	.25
$W_1(1)$	.05	.06	.08	.06	.07	.06	.08	.08	.13	.10	.07	.07	.15	.13
$W_1(2)$	.04	.05	.04	.06	.03	.05	.05	.07	.12	.07	.06	.06	.10	.13
$Wmax_1$	.05	.05	.07	.06	.05	.05	.07	.07	.20	.10	.07	.07	.13	.17
M	.17	.13	.15	.12	.15	.11	.23	.17	.90	.83	.25	.17	.45	.41

Table 2.2: Empirical Size (DGP-0, $\pi_1 \neq \pi_2,\ \rho = \theta = 0$, Nominal Size = 5%)

	sup$F_{1a}(1)$	sup$F_{1a}(2)$	$UDmax_{1a}$	sup $F_{1b}(1)$	sup $F_{1b}(2)$	$UDmax_{1b}$	$W_1(1)$	$W_1(2)$	$Wmax_1$	M
	(A) $T = 150$									
$\pi_1 = 0,\ \pi_2 = -.2$	.10	.08	.09	.07	.08	.10	.10	.08	.09	.26
$\pi_1 = -.3,\ \pi_2 = -.5$	.09	.07	.10	.06	.08	.10	.09	.08	.10	.49
	(B) $T = 240$									
$\pi_1 = 0,\ \pi_2 = -.2$	.09	.07	.08	.05	.08	.09	.09	.08	.08	.19
$\pi_1 = -.3,\ \pi_2 = -.5$	.08	.06	.08	.03	.06	.06	.08	.07	.08	.30

Table 3.1: Empirical Power with One Break and $T = 150$ (DGP-1)

	$\alpha = 0.5$			$\alpha = 0.6$			$\alpha = 0.7$			$\alpha = 0.8$			$\alpha = 0.9$		
	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M
	(A) $\lambda_1^0 = 0.3$														
$\rho = \theta = 0$	1.0	1.0	.99	.99	.99	.89	.92	.93	.56	.58	.59	.23	.17	.18	.07
$\rho = 0.3,\ \theta = 0$	.97	.97	.78	.90	.88	.57	.73	.71	.35	.46	.43	.19	.14	.12	.07
$\rho = 0.5,\ \theta = 0$	.97	.98	.55	.91	.90	.38	.78	.76	.26	.53	.50	.15	.18	.16	.06
$\rho = 0,\ \theta = 0.5$	.97	.96	.83	.92	.90	.68	.80	.78	.44	.54	.50	.24	.18	.17	.09
$\rho = 0,\ \theta = -0.5$	.99	.98	.99	.97	.96	.98	.91	.87	.82	.66	.60	.41	.25	.20	.12
$\rho = 0.3,\ \theta = 0.5$	.95	.94	.62	.86	.83	.48	.73	.69	.30	.48	.44	.16	.17	.13	.08
$\rho = 0.3,\ \theta = -0.5$	1.0	1.0	1.0	.99	.99	.98	.95	.95	.81	.73	.73	.42	.25	.27	.14
	(B) $\lambda_1^0 = 0.5$														
$\rho = \theta = 0$	.99	.99	.81	.96	.96	.56	.81	.81	.28	.52	.52	.10	.19	.20	.06
$\rho = 0.3,\ \theta = 0$	.93	.92	.42	.85	.84	.28	.70	.66	.18	.45	.42	.09	.17	.16	.05
$\rho = 0.5,\ \theta = 0$	.95	.94	.26	.89	.87	.17	.75	.73	.14	.53	.50	.09	.22	.20	.06
$\rho = 0,\ \theta = 0.5$	.95	.94	.51	.89	.87	.37	.75	.72	.23	.50	.47	.12	.21	.19	.07
$\rho = 0,\ \theta = -0.5$	.99	.88	.92	.84	.78	.74	.70	.62	.43	.43	.35	.16	.15	.12	.07
$\rho = 0.3,\ \theta = 0.5$	.92	.90	.34	.86	.81	.25	.73	.69	.17	.49	.44	.10	.20	.18	.06
$\rho = 0.3,\ \theta = -0.5$	.98	.98	.93	.97	.96	.77	.85	.84	.45	.56	.53	.18	.19	.18	.08
	(C) $\lambda_1^0 = 0.7$														
$\rho = \theta = 0$	.92	.91	.29	.78	.78	.14	.63	.63	.08	.44	.43	.05	.19	.20	.04
$\rho = 0.3,\ \theta = 0$	.85	.84	.13	.74	.72	.08	.61	.58	.06	.43	.41	.04	.17	.16	.04
$\rho = 0.5,\ \theta = 0$	.88	.87	.09	.77	.76	.07	.68	.67	.05	.51	.49	.04	.22	.20	.04
$\rho = 0,\ \theta = 0.5$	.87	.85	.17	.77	.75	.12	.62	.60	.07	.45	.43	.05	.18	.16	.05
$\rho = 0,\ \theta = -0.5$	.68	.61	.32	.53	.46	.20	.41	.34	.11	.25	.20	.06	.09	.07	.04
$\rho = 0.3,\ \theta = 0.5$	.86	.83	.11	.76	.73	.08	.65	.60	.06	.48	.44	.05	.20	.17	.03
$\rho = 0.3,\ \theta = -0.5$	.88	.87	.42	.75	.73	.21	.55	.53	.13	.35	.32	.07	.12	.12	.05

Table 3.2: Empirical Power with One Break and $T = 240$ (DGP-1)

	$\alpha = 0.5$			$\alpha = 0.6$			$\alpha = 0.7$			$\alpha = 0.8$			$\alpha = 0.9$		
	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M
	(A) $\lambda_1^0 = 0.3$														
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.98	.94	.94	.74	.39	.39	.21
$\rho = 0.3,\ \theta = 0$	1.0	1.0	1.0	1.0	1.0	.99	.98	.98	.91	.85	.85	.59	.33	.34	.19
$\rho = 0.5,\ \theta = 0$	1.0	1.0	.98	1.0	1.0	.94	.98	.98	.81	.84	.84	.50	.38	.35	.18
$\rho = 0,\ \theta = 0.5$	1.0	1.0	.99	1.0	1.0	.96	.98	.97	.85	.83	.81	.54	.38	.36	.17
$\rho = 0,\ \theta = -0.5$	1.0	1.0	1.0	1.0	1.0	1.0	.98	.99	.97	.91	.93	.74	.40	.53	.22
$\rho = 0.3,\ \theta = 0.5$	1.0	1.0	.95	1.0	.99	.87	.96	.95	.73	.78	.78	.46	.34	.34	.18
$\rho = 0.3,\ \theta = -0.5$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.99	.96	.94	.83	.51	.45	.25
	(B) $\lambda_1^0 = 0.5$														
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	1.0	.98	.99	.99	.79	.84	.84	.37	.38	.37	.12
$\rho = 0.3,\ \theta = 0$	1.0	1.0	.95	.99	.99	.82	.96	.96	.57	.77	.77	.29	.34	.34	.11
$\rho = 0.5,\ \theta = 0$	1.0	1.0	.79	.99	.99	.66	.96	.95	.45	.79	.76	.26	.39	.35	.10
$\rho = 0,\ \theta = 0.5$	1.0	1.0	.92	.99	.99	.79	.96	.95	.55	.76	.74	.29	.38	.36	.10
$\rho = 0,\ \theta = -0.5$	1.0	.99	.99	1.0	.97	.99	.99	.93	.87	.94	.78	.48	.55	.36	.12
$\rho = 0.3,\ \theta = 0.5$	1.0	1.0	.79	.98	.98	.65	.94	.94	.47	.73	.73	.26	.36	.36	.10
$\rho = 0.3,\ \theta = -0.5$	1.0	1.0	1.0	1.0	.99	.99	.98	.97	.88	.86	.83	.47	.39	.32	.12
	(C) $\lambda_1^0 = 0.7$														
$\rho = \theta = 0$	1.0	1.0	.81	.96	.96	.54	.86	.86	.26	.65	.65	.12	.34	.35	.06
$\rho = 0.3,\ \theta = 0$	.98	.98	.47	.92	.92	.30	.81	.82	.17	.63	.63	.09	.32	.33	.06
$\rho = 0.5,\ \theta = 0$	.97	.96	.31	.93	.92	.21	.84	.82	.13	.68	.66	.08	.36	.34	.05
$\rho = 0,\ \theta = 0.5$	.98	.99	.47	.92	.95	.31	.83	.88	.19	.64	.61	.09	.35	.25	.05
$\rho = 0,\ \theta = -0.5$	.94	.97	.81	.83	.91	.60	.75	.82	.28	.55	.63	.14	.26	.33	.05
$\rho = 0.3,\ \theta = 0.5$	.96	.96	.38	.91	.91	.26	.81	.81	.17	.65	.64	.10	.36	.36	.05
$\rho = 0.3,\ \theta = -0.5$	.98	.97	.86	.93	.90	.61	.83	.79	.29	.62	.56	.12	.25	.20	.05

Table 4.1: Empirical Power with One Break and $T = 150$ (DGP-2)

	$\alpha = 0.5$			$\alpha = 0.6$			$\alpha = 0.7$			$\alpha = 0.8$			$\alpha = 0.9$		
	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M
	(A) $\lambda_1^0 = 0.3$														
$\rho = \theta = 0$	.62	.34	.39	.45	.19	.22	.26	.09	.13	.17	.07	.08	.08	.05	.06
$\rho = 0.3,\ \theta = 0$	.46	.19	.21	.34	.13	.12	.19	.08	.09	.14	.06	.07	.08	.06	.06
$\rho = 0.5,\ \theta = 0$	.41	.23	.13	.29	.14	.09	.20	.10	.08	.14	.07	.06	.07	.06	.06
$\rho = 0,\ \theta = 0.5$	.47	.26	.25	.36	.17	.16	.23	.13	.12	.17	.07	.09	.07	.05	.07
$\rho = 0,\ \theta = -0.5$	.32	.17	.47	.23	.13	.30	.17	.10	.18	.14	.09	.11	.09	.06	.07
$\rho = 0.3,\ \theta = 0.5$	.33	.18	.18	.24	.14	.12	.16	.10	.08	.11	.07	.07	.06	.05	.05
$\rho = 0.3,\ \theta = -0.5$	.60	.47	.40	.49	.31	.28	.30	.19	.15	.22	.13	.09	.10	.07	.05
	(B) $\lambda_1^0 = 0.5$														
$\rho = \theta = 0$	.97	.85	.86	.90	.62	.62	.65	.32	.34	.38	.13	.17	.12	.05	.08
$\rho = 0.3,\ \theta = 0$	.84	.53	.52	.70	.32	.34	.46	.17	.23	.26	.07	.13	.09	.05	.07
$\rho = 0.5,\ \theta = 0$	.79	.51	.32	.63	.33	.22	.43	.20	.18	.24	.09	.10	.08	.06	.06
$\rho = 0,\ \theta = 0.5$	.84	.64	.60	.70	.45	.44	.51	.27	.32	.30	.13	.16	.11	.06	.10
$\rho = 0,\ \theta = -0.5$	.81	.67	.95	.70	.51	.84	.57	.37	.53	.38	.22	.27	.15	.10	.11
$\rho = 0.3,\ \theta = 0.5$	.68	.47	.42	.52	.32	.29	.32	.20	.22	.21	.10	.13	.08	.06	.06
$\rho = 0.3,\ \theta = -0.5$	.94	.91	.96	.89	.81	.86	.72	.56	.54	.50	.31	.29	.19	.11	.11
	(C) $\lambda_1^0 = 0.7$														
$\rho = \theta = 0$	1.0	.99	.99	.99	.94	.93	.91	.63	.62	.60	.25	.29	.17	.06	.10
$\rho = 0.3,\ \theta = 0$	.96	.75	.80	.89	.57	.63	.69	.32	.42	.42	.11	.22	.11	.04	.09
$\rho = 0.5,\ \theta = 0$	.93	.74	.59	.86	.55	.44	.65	.32	.30	.39	.14	.18	.11	.06	.07
$\rho = 0,\ \theta = 0.5$	.98	.83	.86	.93	.69	.73	.80	.45	.52	.56	.23	.27	.23	.07	.12
$\rho = 0,\ \theta = -0.5$	.98	.95	1.0	.95	.90	.98	.88	.77	.86	.66	.48	.50	.24	.17	.16
$\rho = 0.3,\ \theta = 0.5$	.85	.67	.67	.71	.49	.49	.49	.32	.35	.29	.16	.20	.11	.06	.08
$\rho = 0.3,\ \theta = -0.5$	.99	.99	1.0	.99	.98	.99	.92	.88	.87	.74	.57	.52	.26	.17	.15

Table 4.2: Empirical Power with One Break and $T = 240$ (DGP-2)

	$\alpha = 0.5$			$\alpha = 0.6$			$\alpha = 0.7$			$\alpha = 0.8$			$\alpha = 0.9$		
	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M
	(A) $\lambda_1^0 = 0.3$														
$\rho = \theta = 0$	.95	.82	.90	.83	.56	.68	.62	.32	.42	.33	.14	.20	.11	.07	.08
$\rho = 0.3,\ \theta = 0$	.89	.67	.55	.72	.42	.43	.54	.24	.29	.30	.12	.17	.10	.06	.09
$\rho = 0.5,\ \theta = 0$	.82	.55	.38	.68	.33	.31	.50	.20	.23	.28	.10	.15	.10	.07	.08
$\rho = 0,\ \theta = 0.5$	.86	.62	.57	.71	.42	.43	.54	.25	.29	.32	.13	.15	.12	.06	.07
$\rho = 0,\ \theta = -0.5$	.73	.54	.92	.59	.38	.73	.43	.26	.49	.27	.15	.22	.16	.07	.09
$\rho = 0.3,\ \theta = 0.5$	.75	.49	.76	.59	.33	.66	.40	.20	.55	.24	.12	.41	.10	.06	.27
$\rho = 0.3,\ \theta = -0.5$	.89	.75	.77	.80	.60	.60	.62	.40	.44	.40	.23	.18	.18	.08	.07
	(B) $\lambda_1^0 = 0.5$														
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	.98	.98	.96	.82	.88	.69	.37	.50	.22	.09	.14
$\rho = 0.3,\ \theta = 0$	1.0	.96	.97	.97	.85	.86	.90	.62	.69	.60	.27	.41	.21	.07	.15
$\rho = 0.5,\ \theta = 0$	.99	.93	.85	.96	.76	.74	.86	.52	.57	.57	.24	.34	.21	.07	.16
$\rho = 0,\ \theta = 0.5$	1.0	.95	.94	.98	.82	.82	.90	.60	.67	.63	.30	.39	.28	.09	.13
$\rho = 0,\ \theta = -0.5$	.98	.93	1.0	.95	.87	.99	.88	.76	.90	.68	.48	.55	.34	.18	.15
$\rho = 0.3,\ \theta = 0.5$	.98	.91	.84	.93	.73	.71	.78	.50	.55	.47	.24	.35	.19	.08	.14
$\rho = 0.3,\ \theta = -0.5$	.99	.98	1.0	.99	.95	.99	.95	.85	.91	.77	.57	.59	.36	.16	.16
	(C) $\lambda_1^0 = 0.7$														
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.99	.99	.94	.71	.81	.38	.11	.24
$\rho = 0.3,\ \theta = 0$	1.0	1.0	1.0	1.0	.98	.99	.99	.89	.94	.87	.52	.70	.32	.10	.25
$\rho = 0.5,\ \theta = 0$	1.0	.99	1.0	1.0	.97	.96	.99	.81	.87	.83	.43	.60	.33	.09	.22
$\rho = 0,\ \theta = 0.5$	1.0	1.0	.99	1.0	.96	.96	.99	.84	.89	.83	.50	.61	.39	.12	.19
$\rho = 0,\ \theta = -0.5$	1.0	.99	1.0	1.0	.99	1.0	.99	.96	.99	.92	.83	.75	.56	.37	.24
$\rho = 0.3,\ \theta = 0.5$	1.0	.99	.96	.99	.93	.90	.94	.76	.77	.72	.42	.53	.27	.09	.20
$\rho = 0.3,\ \theta = -0.5$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.98	1.0	.94	.85	.87	.54	.33	.28

Table 5.1: Empirical Power with One Break (DGP-3)

	$\alpha = 0.5$			$\alpha = 0.6$			$\alpha = 0.7$			$\alpha = 0.8$			$\alpha = 0.9$		
	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M
	(A) $\lambda_1^0 = 0.5,\ T = 150$														
$\pi_1 = 0,\ \pi_2 = -.2$	1.0	1.0	.88	.97	.97	.74	.89	.89	.51	.65	.65	.28	.31	.31	.13
$\pi_1 = -.3,\ \pi_2 = -.5$	.98	.98	.71	.90	.89	.65	.73	.73	.55	.42	.42	.40	.18	.17	.27
	(B) $\lambda_1^0 = 0.5,\ T = 240$														
$\pi_1 = 0,\ \pi_2 = -.2$	1.0	1.0	.94	1.0	1.0	.88	.99	.99	.73	.91	.90	.51	.55	.52	.19
$\pi_1 = -.3,\ \pi_2 = -.5$	1.0	1.0	.79	.99	.99	.66	.97	.97	.48	.79	.78	.35	.38	.34	.21

Table 5.2: Empirical Power with One Break (DGP-4)

	$\alpha = 0.5$			$\alpha = 0.6$			$\alpha = 0.7$			$\alpha = 0.8$			$\alpha = 0.9$		
	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M	$\sup F_{1b}(1)$	$W_1(1)$	M
	(A) $\lambda_1^0 = 0.5,\ T = 150$														
$\pi_1 = 0,\ \pi_2 = -.2$	.82	.66	.88	.72	.50	.70	.55	.29	.43	.30	.17	.23	.12	.12	.14
$\pi_1 = -.3,\ \pi_2 = -.5$	.60	.29	.35	.39	.15	.27	.23	.08	.19	.13	.07	.12	.05	.05	.07
	(B) $\lambda_1^0 = 0.5,\ T = 240$														
$\pi_1 = 0,\ \pi_2 = -.2$	.98	.91	1.0	.94	.77	1.0	.79	.54	.90	.53	.31	.57	.18	.13	.23
$\pi_1 = -.3,\ \pi_2 = -.5$	.95	.74	.79	.81	.50	.66	.52	.21	.45	.25	.09	.26	.06	.05	.10

Table 6: Empirical Power with One Break (DGP-5)

	$\alpha = 0.5$			$\alpha = 0.6$			$\alpha = 0.7$			$\alpha = 0.8$			$\alpha = 0.9$		
	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M	$\sup F_{1a}(1)$	$W_1(1)$	M
	(A) $\lambda_1^0 = 0.5,\ T = 150$														
$\rho = \theta = 0$	.99	.99	.90	.94	.94	.68	.76	.77	.39	.41	.40	.17	.15	.16	.09
$\rho = 0.3,\ \theta = 0$	.88	.86	.53	.75	.72	.39	.52	.49	.25	.29	.26	.14	.11	.11	.07
$\rho = 0.5,\ \theta = 0$	.89	.88	.35	.78	.75	.27	.57	.55	.19	.36	.32	.12	.14	.13	.07
$\rho = 0,\ \theta = 0.5$	.91	.89	.64	.82	.79	.48	.65	.59	.32	.38	.35	.18	.15	.14	.09
$\rho = 0,\ \theta = -0.5$	.89	.88	.98	.81	.78	.87	.70	.62	.60	.47	.35	.28	.20	.12	.13
$\rho = 0.3,\ \theta = 0.5$	.84	.77	.44	.72	.65	.33	.52	.47	.24	.31	.27	.14	.13	.12	.07
$\rho = 0.3,\ \theta = -0.5$	.97	.97	.97	.94	.93	.88	.83	.82	.61	.59	.56	.30	.24	.22	.13
	(B) $\lambda_1^0 = 0.5,\ T = 240$														
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	1.0	.99	.98	.98	.91	.82	.81	.53	.30	.29	.19
$\rho = 0.3,\ \theta = 0$	1.0	1.0	.98	.99	.98	.89	.93	.93	.72	.69	.68	.42	.23	.23	.17
$\rho = 0.5,\ \theta = 0$	1.0	1.0	.88	.98	.97	.77	.92	.91	.59	.69	.66	.36	.26	.23	.15
$\rho = 0,\ \theta = 0.5$	1.0	1.0	.94	.97	.97	.85	.89	.89	.70	.64	.65	.41	.25	.26	.16
$\rho = 0,\ \theta = -0.5$	1.0	.98	1.0	1.0	.96	.99	.99	.89	.92	.94	.74	.60	.55	.37	.18
$\rho = 0.3,\ \theta = 0.5$	.99	.99	.85	.96	.95	.72	.88	.87	.58	.61	.61	.35	.25	.25	.16
$\rho = 0.3,\ \theta = -0.5$	1.0	1.0	1.0	.99	.98	.99	.97	.94	.95	.85	.79	.64	.42	.33	.19

Table 7.1: Empirical Power with Two Breaks and $T = 150$ (DGP-6)

	$\alpha = 0.5$				$\alpha = 0.6$				$\alpha = 0.7$				$\alpha = 0.8$				$\alpha = 0.9$			
	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M
	(A) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.6$																			
$\rho = \theta = 0$	.80	.80	.81	.40	.65	.64	.64	.22	.43	.41	.42	.14	.26	.25	.26	.08	.10	.10	.10	.05
$\rho = 0.3,\ \theta = 0$	.74	.69	.68	.18	.62	.56	.55	.12	.42	.37	.35	.08	.23	.20	.19	.07	.08	.07	.07	.05
$\rho = 0.5,\ \theta = 0$	.77	.74	.73	.13	.66	.63	.62	.09	.46	.43	.44	.07	.27	.25	.25	.07	.10	.08	.07	.04
$\rho = 0,\ \theta = 0.5$	.74	.71	.71	.25	.64	.59	.58	.17	.43	.38	.38	.12	.24	.20	.20	.09	.07	.06	.05	.06
$\rho = 0,\ \theta = -0.5$	.56	.50	.44	.52	.44	.37	.33	.35	.30	.24	.20	.21	.17	.13	.12	.12	.06	.06	.05	.06
$\rho = 0.3,\ \theta = 0.5$	.73	.70	.71	.18	.63	.62	.62	.12	.46	.42	.42	.09	.25	.23	.23	.07	.07	.10	.07	.05
$\rho = 0.3,\ \theta = -0.5$	.80	.79	.81	.57	.66	.63	.65	.34	.44	.40	.42	.21	.23	.21	.22	.12	.10	.09	.10	.06
	(B) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	.94	.94	.94	.68	.79	.78	.78	.43	.56	.54	.54	.22	.32	.31	.31	.12	.11	.10	.10	.05
$\rho = 0.3,\ \theta = 0$	.83	.80	.79	.35	.71	.65	.63	.22	.50	.45	.42	.15	.28	.23	.22	.10	.09	.07	.07	.05
$\rho = 0.5,\ \theta = 0$	.83	.82	.82	.23	.72	.69	.69	.16	.54	.52	.52	.11	.33	.31	.30	.08	.11	.09	.09	.04
$\rho = 0,\ \theta = 0.5$	.85	.81	.81	.45	.73	.70	.70	.30	.53	.47	.47	.19	.30	.26	.24	.11	.09	.07	.07	.06
$\rho = 0,\ \theta = -0.5$	.79	.70	.66	.83	.65	.55	.51	.63	.46	.37	.34	.39	.24	.18	.18	.19	.08	.07	.07	.08
$\rho = 0.3,\ \theta = 0.5$	.81	.79	.83	.29	.70	.67	.72	.21	.52	.47	.54	.14	.31	.28	.34	.09	.11	.09	.13	.05
$\rho = 0.3,\ \theta = -0.5$	.92	.92	.94	.86	.84	.84	.85	.63	.61	.60	.62	.38	.34	.32	.33	.19	.12	.11	.12	.09
	(C) $\lambda_1^0 = 0.4,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	.83	.82	.82	.37	.69	.68	.69	.24	.45	.45	.45	.12	.26	.27	.27	.08	.09	.09	.10	.05
$\rho = 0.3,\ \theta = 0$	.78	.78	.78	.20	.68	.68	.67	.13	.48	.48	.48	.09	.29	.31	.30	.07	.09	.11	.12	.04
$\rho = 0.5,\ \theta = 0$	.78	.78	.78	.12	.69	.69	.69	.10	.53	.52	.52	.07	.34	.34	.34	.06	.11	.12	.12	.04
$\rho = 0,\ \theta = 0.5$	.80	.80	.81	.25	.70	.70	.70	.18	.52	.52	.53	.12	.32	.34	.34	.07	.11	.12	.12	.05
$\rho = 0,\ \theta = -0.5$	.62	.50	.48	.53	.45	.37	.34	.34	.31	.23	.21	.18	.18	.12	.12	.12	.06	.05	.05	.06
$\rho = 0.3,\ \theta = 0.5$	.78	.79	.79	.16	.69	.68	.68	.12	.52	.54	.54	.10	.33	.34	.35	.07	.12	.13	.13	.04
$\rho = 0.3,\ \theta = -0.5$	.84	.84	.86	.57	.70	.69	.69	.36	.47	.45	.45	.20	.25	.24	.26	.12	.09	.09	.09	.06

Table 7.2: Empirical Power with Two Breaks and $T = 240$ (DGP-6)

	$\alpha = 0.5$				$\alpha = 0.6$				$\alpha = 0.7$				$\alpha = 0.8$				$\alpha = 0.9$			
	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M
	(A) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.6$																			
$\rho = \theta = 0$	.96	.96	.96	.90	.89	.88	.88	.68	.73	.70	.71	.39	.43	.41	.41	.20	.16	.14	.14	.08
$\rho = 0.3,\ \theta = 0$	.94	.92	.93	.59	.84	.80	.83	.41	.69	.64	.67	.28	.42	.39	.42	.16	.13	.12	.13	.09
$\rho = 0.5,\ \theta = 0$	.92	.91	.92	.43	.84	.82	.82	.30	.70	.68	.68	.22	.47	.46	.46	.14	.16	.15	.14	.10
$\rho = 0,\ \theta = 0.5$	.92	.90	.91	.61	.84	.81	.81	.41	.66	.62	.63	.28	.41	.39	.40	.15	.14	.13	.13	.08
$\rho = 0,\ \theta = -0.5$	.86	.85	.83	.92	.74	.73	.70	.75	.60	.58	.54	.49	.38	.38	.32	.22	.13	.13	.10	.08
$\rho = 0.3,\ \theta = 0.5$	.89	.88	.89	.48	.79	.77	.77	.35	.65	.63	.63	.26	.42	.41	.42	.14	.14	.13	.13	.09
$\rho = 0.3,\ \theta = -0.5$	.94	.94	.95	.94	.86	.85	.86	.76	.70	.69	.71	.48	.43	.42	.43	.20	.14	.13	.13	.08
	(B) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	.98	.98	.98	.98	.85	.85	.85	.88	.58	.56	.56	.57	.20	.18	.18	.27
$\rho = 0.3,\ \theta = 0$	.98	.98	.98	.86	.94	.92	.94	.69	.79	.75	.78	.48	.51	.48	.51	.25	.18	.15	.17	.10
$\rho = 0.5,\ \theta = 0$	.98	.97	.97	.68	.93	.91	.92	.54	.80	.77	.77	.38	.56	.53	.53	.22	.20	.19	.19	.11
$\rho = 0,\ \theta = 0.5$	.98	.97	.98	.84	.92	.91	.91	.68	.77	.73	.74	.48	.51	.48	.49	.26	.18	.16	.17	.10
$\rho = 0,\ \theta = -0.5$	.95	.95	.94	.99	.90	.90	.88	.95	.78	.77	.74	.77	.55	.55	.50	.38	.20	.20	.16	.13
$\rho = 0.3,\ \theta = 0.5$	.95	.95	.95	.71	.88	.87	.87	.57	.74	.73	.72	.41	.49	.48	.48	.23	.17	.17	.17	.11
$\rho = 0.3,\ \theta = -0.5$	.99	.99	.99	1.0	.97	.97	.97	.97	.85	.85	.86	.77	.60	.60	.62	.38	.19	.19	.20	.12
	(C) $\lambda_1^0 = 0.4,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	.97	.97	.97	.88	.92	.91	.88	.66	.77	.75	.71	.39	.49	.48	.41	.17	.18	.19	.14	.08
$\rho = 0.3,\ \theta = 0$	.95	.93	.93	.58	.85	.83	.83	.42	.73	.69	.67	.26	.48	.43	.42	.15	.17	.15	.13	.08
$\rho = 0.5,\ \theta = 0$	.95	.94	.92	.40	.86	.84	.82	.29	.75	.73	.68	.22	.52	.50	.46	.13	.20	.19	.14	.07
$\rho = 0,\ \theta = 0.5$	.94	.93	.91	.59	.84	.81	.81	.43	.73	.71	.63	.28	.48	.45	.40	.13	.16	.16	.13	.06
$\rho = 0,\ \theta = -0.5$	.87	.87	.83	.91	.80	.79	.70	.74	.63	.63	.54	.49	.42	.41	.32	.19	.14	.14	.10	.08
$\rho = 0.3,\ \theta = 0.5$	.92	.91	.89	.73	.82	.80	.77	.65	.71	.70	.63	.53	.48	.47	.42	.38	.17	.16	.13	.26
$\rho = 0.3,\ \theta = -0.5$	.94	.95	.95	.93	.88	.88	.86	.75	.73	.73	.71	.46	.45	.46	.43	.18	.15	.15	.13	.08

Table 8.1: Empirical Power with Two Breaks and $T = 150$ (DGP-7)

	$\alpha = 0.5$				$\alpha = 0.6$				$\alpha = 0.7$				$\alpha = 0.8$				$\alpha = 0.9$			
	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M
	(A) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.6$																			
$\rho = \theta = 0$	.98	.97	.87	.77	.92	.86	.69	.50	.69	.61	.47	.27	.36	.30	.22	.14	.11	.10	.08	.07
$\rho = 0.3,\ \theta = 0$	.84	.73	.62	.41	.72	.58	.49	.28	.53	.41	.37	.19	.27	.17	.17	.10	.07	.06	.06	.08
$\rho = 0.5,\ \theta = 0$	.84	.78	.69	.64	.74	.65	.58	.51	.58	.51	.46	.46	.33	.27	.24	.35	.10	.08	.09	.25
$\rho = 0,\ \theta = 0.5$	.87	.81	.66	.49	.80	.70	.53	.37	.61	.51	.39	.24	.36	.25	.17	.14	.11	.08	.07	.09
$\rho = 0,\ \theta = -0.5$	.85	.86	.84	.85	.77	.78	.72	.61	.56	.57	.45	.41	.38	.35	.24	.19	.15	.14	.10	.09
$\rho = 0.3,\ \theta = 0.5$	.81	.75	.68	.35	.71	.63	.56	.24	.56	.50	.44	.18	.32	.26	.24	.11	.10	.08	.08	.07
$\rho = 0.3,\ \theta = -0.5$	.98	.98	.91	.90	.97	.92	.77	.71	.83	.70	.50	.46	.50	.35	.22	.23	.16	.14	.11	.10
	(B) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	.96	.93	.78	.57	.86	.78	.60	.32	.62	.54	.41	.17	.36	.31	.23	.10	.13	.11	.09	.06
$\rho = 0.3,\ \theta = 0$	.82	.71	.61	.29	.71	.59	.51	.17	.52	.41	.38	.13	.31	.22	.21	.08	.10	.07	.08	.06
$\rho = 0.5,\ \theta = 0$	.84	.77	.69	.18	.74	.67	.60	.13	.57	.51	.47	.10	.37	.32	.29	.08	.12	.11	.10	.06
$\rho = 0,\ \theta = 0.5$	.84	.78	.70	.37	.77	.66	.59	.24	.58	.48	.45	.16	.37	.27	.26	.11	.13	.08	.12	.08
$\rho = 0,\ \theta = -0.5$	.73	.78	.72	.65	.60	.65	.53	.42	.41	.41	.30	.25	.27	.25	.16	.13	.11	.10	.08	.07
$\rho = 0.3,\ \theta = 0.5$	.81	.75	.72	.25	.72	.65	.62	.16	.56	.50	.50	.12	.35	.28	.30	.08	.11	.09	.13	.05
$\rho = 0.3,\ \theta = -0.5$	.97	.94	.80	.63	.92	.82	.61	.42	.73	.55	.34	.22	.44	.29	.15	.11	.15	.12	.07	.08
	(C) $\lambda_1^0 = 0.4,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	1.0	.99	.82	.77	.95	.88	.59	.52	.76	.64	.35	.28	.44	.34	.18	.15	.15	.12	.06	.08
$\rho = 0.3,\ \theta = 0$	.93	.89	.66	.43	.84	.76	.50	.29	.65	.56	.36	.20	.44	.35	.21	.11	.15	.13	.08	.07
$\rho = 0.5,\ \theta = 0$	.93	.88	.69	.27	.83	.75	.57	.21	.66	.58	.42	.15	.45	.36	.25	.09	.15	.13	.09	.07
$\rho = 0,\ \theta = 0.5$	.96	.92	.72	.52	.88	.83	.56	.37	.73	.66	.41	.26	.50	.45	.23	.14	.20	.20	.10	.10
$\rho = 0,\ \theta = -0.5$	.87	.87	.83	1.0	.76	.78	.71	.97	.59	.58	.45	.83	.38	.35	.23	.47	.14	.14	.10	.12
$\rho = 0.3,\ \theta = 0.5$	.91	.88	.72	.35	.82	.79	.58	.25	.67	.63	.44	.19	.48	.44	.26	.11	.20	.19	.10	.07
$\rho = 0.3,\ \theta = -0.5$	.99	.99	.91	.91	.96	.92	.72	.74	.83	.70	.41	.46	.53	.38	.19	.23	.17	.15	.09	.10

Table 8.2: Empirical Power with Two Breaks and $T = 240$ (DGP-7)

	$\alpha = 0.5$				$\alpha = 0.6$				$\alpha = 0.7$				$\alpha = 0.8$				$\alpha = 0.9$			
	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M
	(A) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.6$																			
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	1.0	1.0	.96	.95	.98	.96	.81	.75	.83	.71	.50	.40	.29	.21	.18	.13
$\rho = 0.3,\ \theta = 0$	1.0	.99	.95	.92	.98	.94	.81	.78	.93	.85	.69	.56	.75	.59	.46	.32	.27	.18	.16	.14
$\rho = 0.5,\ \theta = 0$	1.0	.99	.94	.73	.98	.94	.82	.60	.93	.85	.71	.45	.75	.61	.51	.28	.29	.21	.19	.12
$\rho = 0,\ \theta = 0.5$	1.0	.98	.90	.91	.97	.93	.78	.79	.90	.82	.64	.57	.70	.59	.43	.30	.29	.19	.15	.12
$\rho = 0,\ \theta = -0.5$	1.0	.99	.97	1.0	.99	.95	.91	.97	.97	.88	.78	.82	.82	.60	.48	.45	.38	.18	.16	.12
$\rho = 0.3,\ \theta = 0.5$	.99	.97	.87	.78	.96	.90	.75	.66	.88	.78	.64	.50	.66	.55	.43	.29	.28	.18	.16	.12
$\rho = 0.3,\ \theta = -0.5$	1.0	1.0	.99	1.0	1.0	.99	.97	.98	.98	.97	.89	.82	.88	.75	.58	.46	.37	.23	.17	.13
	(B) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	1.0	1.0	.98	1.0	1.0	.98	.86	.97	.95	.89	.66	.83	.75	.60	.41	.53	.29	.21	.18	.22
$\rho = 0.3,\ \theta = 0$	.99	.98	.89	.76	.96	.90	.73	.60	.90	.77	.60	.41	.69	.54	.41	.23	.29	.19	.18	.10
$\rho = 0.5,\ \theta = 0$	.99	.98	.89	.55	.96	.89	.74	.43	.90	.78	.65	.32	.70	.58	.47	.20	.30	.22	.19	.09
$\rho = 0,\ \theta = 0.5$	.99	.97	.86	.77	.96	.89	.71	.60	.87	.75	.58	.41	.66	.54	.40	.22	.29	.20	.15	.09
$\rho = 0,\ \theta = -0.5$	.98	.94	.88	.99	.95	.87	.76	.90	.90	.76	.60	.63	.71	.46	.38	.31	.29	.15	.12	.09
$\rho = 0.3,\ \theta = 0.5$	.98	.94	.84	.66	.93	.85	.71	.50	.83	.72	.59	.37	.64	.53	.42	.20	.30	.22	.18	.10
$\rho = 0.3,\ \theta = -0.5$	1.0	.99	.98	.99	.98	.97	.91	.89	.96	.89	.74	.63	.80	.64	.43	.32	.31	.19	.12	.09
	(C) $\lambda_1^0 = 0.4,\ \lambda_2^0 = 0.7$																			
$\rho = \theta = 0$	1.0	1.0	1.0	1.0	1.0	1.0	.93	.97	.98	.95	.70	.79	.80	.67	.39	.44	.30	.20	.14	.14
$\rho = 0.3,\ \theta = 0$	1.0	1.0	.93	.92	.99	.96	.81	.79	.94	.96	.62	.60	.71	.58	.38	.37	.28	.18	.16	.14
$\rho = 0.5,\ \theta = 0$	1.0	1.0	.90	.77	.98	.95	.79	.62	.94	.83	.64	.49	.71	.61	.44	.31	.31	.21	.17	.13
$\rho = 0,\ \theta = 0.5$	1.0	.98	.91	.91	.98	.94	.80	.76	.90	.81	.64	.61	.69	.56	.41	.35	.29	.19	.17	.12
$\rho = 0,\ \theta = -0.5$	.99	.98	.95	1.0	.99	.96	.90	.97	.96	.86	.72	.83	.80	.58	.41	.47	.36	.19	.15	.12
$\rho = 0.3,\ \theta = 0.5$	.99	.98	.86	.96	.97	.94	.75	.89	.91	.83	.61	.82	.69	.61	.41	.63	.31	.26	.18	.35
$\rho = 0.3,\ \theta = -0.5$	1.0	1.0	.99	1.0	1.0	.99	.96	.98	.98	.96	.85	.84	.86	.75	.49	.48	.37	.25	.15	.13

Table 9.1: Empirical Power with Two Breaks (DGP-8)

	$\alpha=0.5$				$\alpha=0.6$				$\alpha=0.7$				$\alpha=0.8$				$\alpha=0.9$			
	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M
	(A) $\lambda_1^0=0.3,\ \lambda_1^0=0.6,\ T=150$																			
$\pi_1=0,\ \pi_2=-.2$	.94	.94	.94	.61	.85	.84	.85	.46	.70	.69	.69	.27	.44	.42	.42	.16	.20	.20	.19	.09
$\pi_1=-.3,\ \pi_2=-.5$	.95	.94	.94	.52	.84	.84	.85	.49	.69	.68	.69	.36	.51	.52	.53	.27	.29	.29	.32	.20
	(B) $\lambda_1^0=0.3,\ \lambda_1^0=0.6,\ T=240$																			
$\pi_1=0,\ \pi_2=-.2$	.99	.99	1.0	.74	.98	.97	.97	.66	.88	.87	.88	.49	.69	.65	.66	.29	.31	.30	.30	.13
$\pi_1=-.3,\ \pi_2=-.5$	.99	1.0	1.0	.46	.97	.97	.97	.37	.85	.84	.85	.32	.65	.63	.64	.23	.34	.33	.34	.17

Table 9.2: Empirical Power with Two Breaks (DGP-9)

	$\alpha=0.5$				$\alpha=0.6$				$\alpha=0.7$				$\alpha=0.8$				$\alpha=0.9$			
	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M	$\sup F_{1b}(2)$	UD_{1b}	wm_1	M
	(A) $\lambda_1^0=0.3,\ \lambda_1^0=0.6,\ T=150$																			
$\pi_1=0,\ \pi_2=-.2$	.96	.95	.83	.79	.86	.84	.65	.56	.64	.57	.41	.36	.30	.32	.24	.22	.09	.14	.14	.12
$\pi_1=-.3,\ \pi_2=-.5$	.79	.82	.65	.77	.69	.73	.52	.63	.51	.58	.40	.51	.34	.45	.33	.36	.14	.29	.26	.24
	(B) $\lambda_1^0=0.3,\ \lambda_1^0=0.6,\ T=240$																			
$\pi_1=0,\ \pi_2=-.2$	1.0	1.0	.98	1.0	.99	.98	.91	.97	.96	.92	.75	.79	.73	.63	.42	.49	.21	.22	.19	.21
$\pi_1=-.3,\ \pi_2=-.5$	.99	.96	.75	.99	.94	.88	.59	.96	.80	.75	.50	.84	.50	.49	.31	.54	.19	.33	.22	.29

Table 10: Empirical Power with Two Breaks (DGP-10)

	$\alpha = 0.5$				$\alpha = 0.6$				$\alpha = 0.7$				$\alpha = 0.8$				$\alpha = 0.9$			
	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M	$\sup F_{1a}(2)$	UD_{1a}	wm_1	M
	(A) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.6,\ T = 150$																			
$\rho = \theta = 0$	.64	.64	.66	.54	.46	.44	.45	.33	.28	.27	.28	.22	.15	.15	.16	.12	.07	.08	.08	.06
$\rho = 0.3,\ \theta = 0$	.48	.40	.37	.27	.33	.25	.24	.19	.19	.15	.15	.14	.10	.08	.07	.12	.06	.06	.06	.06
$\rho = 0.5,\ \theta = 0$	.44	.38	.39	.19	.31	.27	.27	.14	.20	.17	.18	.11	.10	.09	.10	.10	.07	.06	.06	.06
$\rho = 0,\ \theta = 0.5$	.51	.45	.44	.33	.37	.32	.32	.26	.23	.20	.21	.20	.13	.11	.12	.13	.08	.06	.06	.08
$\rho = 0,\ \theta = -0.5$	.46	.45	.41	.68	.31	.32	.31	.48	.25	.25	.24	.31	.16	.17	.16	.17	.08	.08	.08	.08
$\rho = 0.3,\ \theta = 0.5$	.41	.39	.39	.25	.29	.28	.29	.17	.21	.18	.19	.16	.10	.09	.10	.11	.08	.07	.07	.06
$\rho = 0.3,\ \theta = -0.5$	.90	.90	.74	.73	.81	.81	.58	.51	.62	.63	.39	.32	.38	.38	.24	.19	.16	.15	.11	.09
	(B) $\lambda_1^0 = 0.3,\ \lambda_2^0 = 0.6,\ T = 240$																			
$\rho = \theta = 0$	.95	.95	.95	.96	.82	.79	.81	.84	.60	.57	.58	.56	.28	.25	.26	.30	.10	.09	.09	.13
$\rho = 0.3,\ \theta = 0$	.86	.82	.85	.74	.68	.62	.67	.55	.46	.41	.46	.39	.23	.18	.21	.24	.09	.07	.09	.13
$\rho = 0.5,\ \theta = 0$	.93	1.0	.77	.55	.86	.98	.59	.43	.69	.90	.41	.32	.43	.59	.21	.21	.14	.17	.09	.12
$\rho = 0,\ \theta = 0.5$	.83	.80	.81	.73	.67	.63	.64	.54	.46	.41	.42	.40	.24	.20	.22	.23	.10	.09	.10	.11
$\rho = 0,\ \theta = -0.5$	.79	.79	.74	.97	.65	.65	.59	.87	.49	.48	.45	.65	.27	.29	.26	.33	.15	.14	.12	.12
$\rho = 0.3,\ \theta = 0.5$	.71	.68	.68	.59	.54	.51	.52	.46	.37	.34	.34	.36	.19	.17	.17	.23	.08	.08	.08	.13
$\rho = 0.3,\ \theta = -0.5$	.90	.90	.92	.98	.81	.81	.83	.88	.62	.63	.65	.65	.38	.38	.41	.32	.16	.15	.17	.13